Philadelphia University
Department of Basic Sciences

Final Exam **Linear Algebra 2** **15–6–2006**

Each problem is worth 8 points. Two points extra for turning your mobile phone off.

1. Let $A \in M_{n\times n}(F)$.
   (a) Let c be an eigenvalue of A. Prove that the set of all eigenvectors of A corresponding to the eigenvalue c is a subspace of F^n.
   (b) Suppose $\det A \neq 0$. Prove that the matrix AB is similar to BA for all $B \in M_{n\times n}(F)$.

2. Let $A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 3 & 1 \\ 3 & 1 & 3 \end{bmatrix}$
   (a) Write A as a product of elementary matrices $E_1, E_2, \ldots, E_n$.
   (b) Evaluate $\det A$ by computing $\det E_1, \det E_2, \ldots, \det E_n$.

3. Let $T(x, y) = (2x + y, x + 2y)$ be a linear operator on R^2.
   (a) Find the matrix of T with respect to the basis $\{(2, 1), (3, 2)\}$.
   (b) Find a basis B' of R^2 such that the matrix of T with respect to B' is diagonal.

4. Suppose the matrix A is diagonalizable such that $P^{-1}AP = D$ where
$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$
   (a) Compute A^{10}.
   (b) Solve the system of differential equations $Y' = AY$.

5. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.
   (a) Find the characteristic polynomial of the matrix A.
   (b) Verify the Cayley-Hamilton Theorem for the matrix A.
   (c) Prove that A is not diagonalizable.

6. Let A be the same matrix in Problem (5).
   (a) Write $A = B + C$ such that $BC = CB$.
   (b) Use the result to evaluate the matrix exponential e^A.
   (c) Solve the system of differential equations $Y' = AY$.

–Amin Witno