1 Complex Numbers

Definition. The set of complex numbers is denoted by

$$\mathbb{C} = \{(a, b) \mid a, b \in \mathbb{R}\}$$

and comes with two binary operations $+$ and $\times$ defined by

$$(a, b) + (c, d) = (a + c, b + d)$$
$$(a, b) \times (c, d) = (ac - bd, ad + bc)$$

Theorem 1.1. $\mathbb{C}$ is a field under these two operations.

Proof. It is clear that addition is commutative with zero element $(0, 0)$ and negative element $-(x, y) = (-x, -y)$. We can check that multiplication is also commutative, with identity $(1, 0)$ and inverse element for $(x, y) \neq (0, 0)$ given by

$$(x, y)^{-1} = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\right)$$

$\triangledown$

Example. We illustrate these operations:
1. $(5, -1) + (4, 2) = (5 + 4, -1 + 2) = (9, 1)$
2. $(8, 3) - (2, 3) = (8 - 2, 3 - 3) = (6, 0)$
3. $(2, -2) \times (-1, 5) = (2(-1) - (-2)5, 2 \cdot 5 + (-2)(-1)) = (8, 12)$
4. $(2, -1) \div (3, 2) = (2, -1) \times (3, 2)^{-1} = (2, -1)(\frac{3}{3^2+2^2}, \frac{-2}{3^2+2^2}) = (\frac{4}{13}, \frac{-7}{13})$

Note that if the second coordinate is zero, then we have

$$(a, 0) + (c, 0) = (a + c, 0) \quad \text{and} \quad (a, 0) \times (c, 0) = (ac, 0)$$

Hence, we have $\mathbb{R} \approx \{(x, 0) \in \mathbb{C} \mid x \in \mathbb{R}\}$, i.e., $\mathbb{R} \subseteq \mathbb{C}$ by the identification $x \in \mathbb{R} \iff (x, 0) \in \mathbb{C}$. In particular, $(0, 0) = 0 \in \mathbb{R}$ and $(1, 0) = 1 \in \mathbb{R}$. Next, observe that for all $y \in \mathbb{R}$, we have $(y, 0)(0, 1) = (0, y)$. So if we let $i = (0, 1) \in \mathbb{C}$, then for all $(x, y) \in \mathbb{C}$

$$(x, y) = (x, 0) + (0, y) = (x, 0) + (y, 0)(0, 1) = x + yi$$

And since $i^2 = (0, 1)(0, 1) = (-1, 0) \iff -1 \in \mathbb{R}$, then we may redefine the set of complex numbers, i.e.,

$$\mathbb{C} = \{x + yi \mid x, y \in \mathbb{R}, i = \sqrt{-1}\}$$

with the *ordinary* addition and multiplication, i.e., we treat i like an indeterminate that is subject to $i^2 = -1$. Obviously, i is not a real number but is an *imaginary* number belonging in $\mathbb{C} - \mathbb{R}$.

Example. We redo the earlier example using the new form:
1. $(5 - i) + (4 + 2i) = 9 + i$
2. $(8 + 3i) - (2 + 3i) = 6$
3. $(2 - 2i) \times (-1 + 5i) = -2 + 10i + 2i - 10i^2 = -2 + 12i - 10(-1) = 8 + 12i$
4. $\frac{2-i}{3+2i} = \frac{2-i}{3+2i} \times \frac{3-2i}{3-2i} = \frac{6-4i-3i+2i^2}{9-6i+6i-4i^2} = \frac{4-7i}{13} = \frac{4}{13} + \frac{-7}{13}\, i$

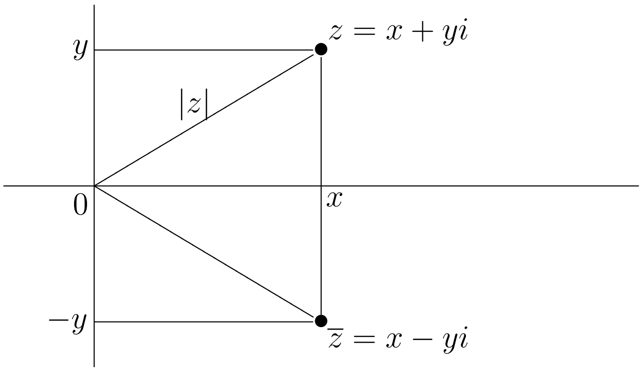
Definition. For every $z = x + yi \in \mathbb{C}$, we define the *conjugate* $\overline{z}$, the *modulus* or *absolute value* $|z|$, the *real part* $\operatorname{Re} z$, and the *imaginary part* $\operatorname{Im} z$ as follow.

$$\overline{z} = x - yi; \quad |z| = \sqrt{x^2 + y^2}; \quad \operatorname{Re} z = x; \quad \operatorname{Im} z = y$$

Theorem 1.2. It is easy to check these identities relating the above quantities

$$|\overline{z}| = |z|; \quad z\overline{z} = |z|^2; \quad z + \overline{z} = 2\operatorname{Re} z; \quad z - \overline{z} = 2i \operatorname{Im} z$$

We can also view a complex number $(x, y) \in \mathbb{C}$ as a vector in the complex plane, where x is the coordinate with respect to the horizontal real axis and y the vertical imaginary axis.



Theorem 1.3. In particular, we easily see that the *triangular inequality* holds:

$$|\ z_1 + z_2\ | \leq |\ z_1\ | + |\ z_2\ |$$

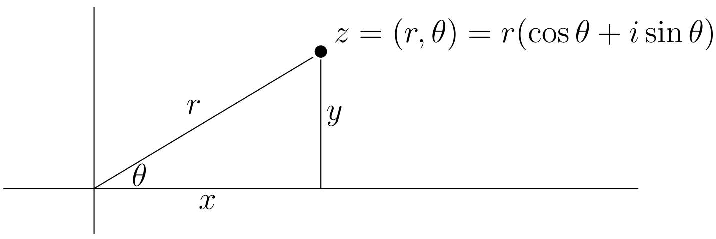
Polar Form

Let θ stand for the angle of the vector (x, y) against the positive real axis measured counter-clockwise, and let $r = |z|$.

Theorem 1.4. The following identities are obvious:

$$x = r\cos\theta; \quad y = r\sin\theta; \quad r = \sqrt{x^2 + y^2}; \quad \tan\theta = \frac{y}{x} \to \theta = \arctan\left(\frac{y}{x}\right)$$

Definition. Every complex number can be represented in its *rectangular form* $z = (x, y) = x + yi$ or in *polar form* $z = (r, \theta) = r(\cos\theta + i\sin\theta)$.



Note that the choice of θ is unique only up to a multiple of 2π. For example, we have $(r, \frac{\pi}{2}) = (r, -\frac{3\pi}{2}) = (r, \frac{5\pi}{2}) = (r, \frac{\pi}{2} + 2\pi n)$ for any $n \in \mathbb{Z}$.

Definition. The *principal argument* of a complex number $z \in \mathbb{C}$ is $\operatorname{Arg} z = \theta \in \mathbb{R}$ with the unique value of $\theta \in (-\pi, \pi]$ such that $z = (r, \theta)$ when expressed in polar form. Moreover, let the *argument* of z mean any of the non-unique value of θ, i.e., $\arg z = \operatorname{Arg} z + 2\pi n$ for any $n \in \mathbb{Z}$.

Example. Express $z = -1 + i$ in polar form.

Solution: $r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$ and $\theta = \arctan(-1) \in (-\pi, \pi] \to \theta = \frac{3\pi}{4}$. Hence, $z = (\sqrt{2}, \frac{3\pi}{4})$ or $z = \sqrt{2}\,(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$.

Theorem 1.5. Note how we obtain conjugate in polar form:

$$z = (r, \theta) \iff \overline{z} = (r, -\theta)$$

Theorem 1.6. Given two complex numbers in polar form, $z_1 = (r, \theta)$ and $z_2 = (s, \psi)$,

$$(r, \theta)(s, \psi) = (rs\,,\ \theta + \psi) \quad \text{and} \quad \frac{(r, \theta)}{(s, \psi)} = \left(\frac{r}{s}\,,\ \theta - \psi\right)$$

Thus $|z_1 z_2| = |z_1|\,|z_2|$ and $\arg(z_1 z_2) = \arg z_1 + \arg z_2$.

Proof. For the first,

$$\begin{aligned}(r, \theta)(s, \psi) &= r(\cos\theta + i\sin\theta) \times s(\cos\psi + i\sin\psi) \\ &= rs\,[(\cos\theta\cos\psi - \sin\theta\sin\psi) + i(\sin\theta\cos\psi + \cos\theta\sin\psi)] \\ &= rs\,[\cos(\theta + \psi) + i\sin(\theta + \psi)]\end{aligned}$$

As for the second, see that $(s, \psi) \times (\frac{r}{s}, \theta - \psi) = (r, \theta)$. ▽

Theorem 1.7. Let $z = (r, \theta)$. Then for all $n \in \mathbb{Z}$,

$$z^{-1} = \left(\tfrac{1}{r}\,, -\theta\right); \quad z^n = (r^n,\ n\theta)\,; \quad z^{\frac{1}{n}} = \left(\sqrt[n]{r}\,,\ \frac{\theta + 2\pi k}{n}\right) \quad \text{where} \quad k \in \{0, 1, \ldots, n-1\}$$

Theorem 1.8. In particular, let $r = 1$ above and we obtain De Moivre's formula

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

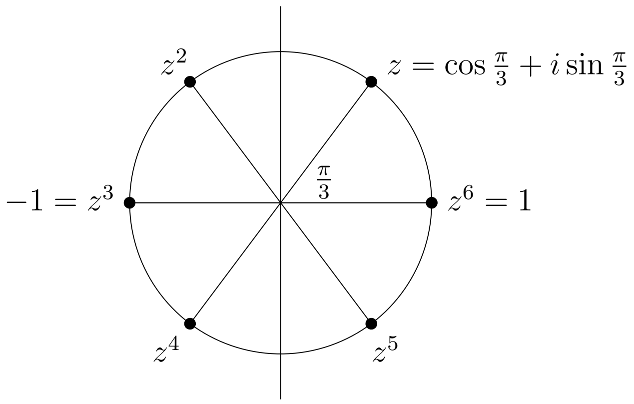
from which, with $n = 2$, we derive the identities $\begin{cases} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ \sin 2\theta &= 2\,\sin\theta\,\cos\theta \end{cases}$

Example. Evaluate $(1 + i)^6$.

Solution: First we convert to polar: $1 + i = (\sqrt{2}, \frac{\pi}{4})$, then use Theorem 1.7: $(1 + i)^6 = (\sqrt{2}^6, \frac{\pi}{4} \times 6) = (8, \frac{3\pi}{2})$, and finally back to rectangular: $(1+i)^6 = 8(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}) = -8i$.

Example. Find all $z \in \mathbb{C}$ such that $z^6 = 1$.

Solution: We have $z = 1^{\frac{1}{6}}$. Because $1 = (1, 0)$ in polar form, then by Theorem 1.7, $z = \left(1, \frac{2\pi k}{6}\right)$ with $k = 0, 1, 2, 3, 4, 5$, equally distributed on the unit circle:



2 Complex Functions

Definition. For all $z = x + yi \in \mathbb{C}$ we define the *complex exponential* function e^z by

$$e^z = e^x \cos y + ie^x \sin y = e^x(\cos y + i \sin y)$$

i.e., when expressed in polar form, $e^{x+yi} = (e^x, y)$.

Example. Evaluate $e^{2+i\pi/2}$.

Solution: We have $z = 2 + i\frac{\pi}{2}$, so $e^z = e^2(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}) = ie^2$.

Example. The following identity is quite well-known:

$$e^{\pi i} = -1$$

Note that if $z = x \in \mathbb{R}$, then $e^z = e^x$, i.e., the complex function e^z is an extension of the real exponential function e^x. Moreover, if $x = 0$ then $e^z = e^{yi} = \cos y + i \sin y$. In general, therefore, $re^{i\theta} = r(\cos\theta + i\sin\theta) = (r, \theta)$.

Definition. We can now express every complex number $z \in \mathbb{C}$ in three different forms: rectangular, polar, and *exponential form*, respectively:

$$z \quad = \quad x + yi \quad = \quad r(\cos\theta + i\sin\theta) \quad = \quad re^{i\theta}$$

Theorem 2.1. The exponential function e^z obeys some familiar properties:

$$e^{z_1}\, e^{z_2} = e^{z_1+z_2} \quad \text{and} \quad \frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2} \quad \text{for all} \quad z_1, z_2 \in \mathbb{C}$$

Since $|e^z| = e^x$, we have $|e^z| > 0$, hence $e^z \neq 0$ for all $z \in \mathbb{C}$, just like the real exponential function e^x. However, beware that e^z can assume a negative real number, e.g., the above $e^{\pi i} = -1$. Moreover, it is easy to see that $e^{z+2\pi i} = e^z$ for all $z \in \mathbb{C}$, showing that unlike the real counterpart, the complex exponential e^z is not a one-to-one function.

Example. Find all $z \in \mathbb{C}$ such that $e^z = 1 + i$.

Solution: Let $z = x + yi$ and $e^z = (e^x, y)$ in polar form. Compare this to $1 + i = (\sqrt{2}, \frac{\pi}{4} + 2\pi n)$. Hence, $x = \ln\sqrt{2}$ and $z = \ln\sqrt{2} + i(\frac{\pi}{4} + 2\pi n)$ for any $n \in \mathbb{Z}$.

Definition. For all $z \in \mathbb{C} - \{0\}$, the *complex logarithms* $\log z$ and $\operatorname{Log} z$ are given by

$$\log z = \ln|z| + i \arg z \quad \text{and} \quad \operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$$

i.e., in terms of r and θ, we have $\log z = \ln r + i(\theta + 2\pi n)$ for any $n \in \mathbb{Z}$. Thus $\log z$ is a multi-valued complex function. To distinguish the two functions, the single-valued $\operatorname{Log} z$ is called the *principal logarithm* function.

Example. Evaluate $\operatorname{Log} z$ and $\log z$ for $z = -1 - i\sqrt{3}$.

Solution: We have $|z| = \sqrt{1+3} = 2$ and $\operatorname{Arg} z = \arctan\sqrt{3} = -\frac{2\pi}{3}$, since we must choose $\theta \in (-\pi, \pi]$. So $\operatorname{Log}(-1-i\sqrt{3}) = \ln 2 - i\frac{2\pi}{3}$ and $\log(-1-i\sqrt{3}) = \ln 2 + i(-\frac{2\pi}{3} + 2\pi n)$.

Note that if $z = x \in \mathbb{R}^+$, then $\operatorname{Log} z = \ln x$, so the function $\operatorname{Log} z$ is an extension of the natural logarithm real function $\ln x = \log_e x$, e.g., we have $\operatorname{Log} 1 = 0$ regardless we treat 1 as a real or as a complex number. (But beware that $\log 1 = 2\pi ni$.) However, in the complex case, we allow logarithms to take on negative real numbers, e.g., $\operatorname{Log}(-1) = \pi i$.

Theorem 2.2. The logarithm function $\log z$ obeys some familiar properties:

$$\log(z_1 z_2) = \log z_1 + \log z_2 \quad \text{and} \quad \log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2 \quad \text{for all} \quad z_1, z_2 \in \mathbb{C}^*$$

The notation $\mathbb{C}^*$ stands for the set of all non-zero complex numbers.

We agree that identities involving the multi-valued $\log z$ are understood as a set of values. The above theorem may not hold for principal logarithm, e.g., $\operatorname{Log}(i^3) \neq 3 \operatorname{Log} i$, as $\operatorname{Log}(i^3) = \operatorname{Log}(-i) = -\frac{\pi}{2}i$ and $3 \operatorname{Log} i = \frac{3\pi}{2}i$.

Theorem 2.3. Here is the relation between e^z and $\log z$ as inverse functions:

$$e^{\log z} = z \quad \text{and} \quad \log(e^z) = z + 2\pi n i$$

Definition. For all $z \in \mathbb{C}^*$, $w \in \mathbb{C}$, the *complex power* z^w is defined by

$$z^w = e^{w \log z}$$

Thus z^w is also muti-valued, unless we specifically state the use of principal logarithm, i.e., the *principal power* $z^w = e^{w \operatorname{Log} z}$. In particular, if $z = x \in \mathbb{R}^+$ and $w \in \mathbb{R}$, then the principal power $z^w = e^{w \ln x} = x^w$ coincides with the real power function.

Example. Evaluate i^{-2i}.

Solution: First we evaluate $\log i = i(\frac{\pi}{2} + 2\pi n)$, then we multiply $(-2i)i(\frac{\pi}{2} + 2\pi n) = \pi + 4\pi n$, and finally $i^{-2i} = e^{\pi + 4\pi n}$ for any value $n \in \mathbb{Z}$.

Theorem 2.4. For all $z, z_1, z_2 \in \mathbb{C}^*$ and $w, w_1, w_2 \in \mathbb{C}$ the following identities hold.

$$(1)\ (z_1 z_2)^w = z_1^w\, z_2^w \quad (2)\ z^{w_1} z^{w_2} = z^{w_1 + w_2} \quad (3)\ \frac{z^{w_1}}{z^{w_2}} = z^{w_1 - w_2} \quad (4)\ (z^w)^{-1} = z^{-w}$$

Trigonometric Functions

Definition. For all $z \in \mathbb{C}$, we define the *complex cosine* and *sine* functions by

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Example. Evaluate $\cos z$ and $\sin z$ for $z = \pi + i$.

Solution: We have $e^{iz} = e^{-1+i\pi} = e^{-1}e^{i\pi} = -\frac{1}{e}$ and $e^{-iz} = e^1 e^{-i\pi} = -e$. Hence $\cos z = \frac{-\frac{1}{e} - e}{2} = -\frac{1+e^2}{2e} \approx -1.54$ and $\sin z = \frac{-\frac{1}{e} + e}{2i} = i\frac{1-e^2}{2e} \approx -1.17i$.

It is clear from their definitions that $\cos z$ is an even function while $\sin z$ is odd, i.e.,

$$\cos(-z) = \cos z \quad \text{and} \quad \sin(-z) = -\sin z$$

just like $\cos x$ and $\sin x$ over $\mathbb{R}$. Moreover, when $z = x \in \mathbb{R}$, then $e^{ix} = \cos x + i \sin x$ and $e^{-ix} = \cos x - i \sin x$, in which case $\cos z = \cos x$ and $\sin z = \sin x$, showing that $\cos z$ and $\sin z$ are extensions of their respective real functions.

Theorem 2.5. The following familiar identities hold for all $z_1, z_2 \in \mathbb{C}$.

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$$
$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$$

which show that both $\cos z$ and $\sin z$ are periodic functions, since

$$\cos(z + 2\pi) = \cos z \quad \text{and} \quad \sin(z + 2\pi) = \sin z$$

Theorem 2.6. Given $z = x + yi$, we can express $\cos z$ and $\sin z$ in rectangular form:

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$
$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

which show that both $\cos z$ and $\sin z$ are unbounded functions, since

$$|\cos z|^2 = \cos^2 x + \sinh^2 y$$
$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$

Proof. Recall that $\cosh y = \frac{e^y+e^{-y}}{2}$ and $\sinh y = \frac{e^y-e^{-y}}{2}$ are both unbounded functions. For the first part, apply Theorem 2.5, together with the identities

$$\cos(iy) = \cosh y \quad \text{and} \quad \sin(iy) = i \sinh y$$

as well as $\cos^2 x + \sin^2 x = 1$ and $\cosh^2 y - \sinh^2 y = 1$ for the second. ▽

Theorem 2.7. The zeros of $\cos z$ and $\sin z$ are all real zeros, i.e.,

$$\cos z = 0 \iff z = \tfrac{\pi}{2} + n\pi \quad \text{and} \quad \sin z = 0 \iff z = n\pi \quad \text{for all} \quad n \in \mathbb{Z}$$

Proof. Since $\cos z = 0$ if and only if $|\cos z| = 0$, Theorem 2.6 gives us that this occurs if and only if $\cos x = 0$ and $\sinh y = 0$. We observe that $\sinh y = 0$ if and only if $y = 0$. Similarly, we also have that $\sin z = 0$ if and only if $\sin x = 0$ and $y = 0$ as claimed. ▽

Definition. Define for all $z \in \mathbb{C}$ the remaining complex trigonometric functions

$$\tan z = \frac{\sin z}{\cos z} \quad \text{and} \quad \cot z = \frac{1}{\tan z} \quad \text{and} \quad \sec z = \frac{1}{\cos z} \quad \text{and} \quad \csc z = \frac{1}{\sin z}$$

Definition. For all $z \in \mathbb{C}$, the *complex hyperbolic* functions are given by

$$\cosh z = \frac{e^z + e^{-z}}{2} \quad \text{and} \quad \sinh z = \frac{e^z - e^{-z}}{2}$$

Theorem 2.8. We have the identities

$$\cos(iz) = \cosh z \quad \text{and} \quad \sin(iz) = i \sinh z$$

3 Limits and Continuity

Definition. The limit of a complex function $f(z)$ at a given point z_0 is defined by

$$\lim_{z \to z_0} f(z) = w$$

such that $\forall \varepsilon > 0, \exists \delta > 0$ with condition that if $0 < |z - z_0| < \delta$ then $|f(z) - w| < \varepsilon$.

Example. Prove that $\lim_{z\to 1} \frac{i\bar{z}}{2} = \frac{i}{2}$.

Solution: Given $\varepsilon > 0$, we let $\delta = 2\varepsilon$. Now if $|z - z_0| < \delta$, i.e., $|z - 1| < 2\varepsilon$, then $\to |\frac{i}{2}| \cdot |z - 1| < |\frac{i}{2}| \cdot 2\varepsilon \to |\frac{i}{2}(z - 1)| < \frac{1}{2} \cdot 2\varepsilon \to |\frac{iz}{2} - \frac{i}{2}| < \varepsilon$, i.e., $|f(z) - w| < \varepsilon$.

Example. Prove that $\lim_{z\to(2+i)} 3z - iz = 7 + i$.

Solution: Given $\varepsilon > 0$, we let $\delta = \frac{\varepsilon}{4}$. Now if $|z - z_0| < \delta$, i.e., $|z - (2 + i)| < \frac{\varepsilon}{4}$ (*)

(a) multiply (*) by 3 $\to |3z - (6 + 3i)| < \frac{3\varepsilon}{4}$

(b) multiply (*) by $|-i| = 1 \to |-iz - (-2i + 1)| < 1 \cdot \frac{\varepsilon}{4}$

then add (a) and (b) $\to |3z - (6 + 3i)| + |-iz - (-2i + 1)| < \frac{3\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon$.

By triangle inequality $\to |3z-(6+3i)-iz-(-2i+1)| < \varepsilon \to |(3z-iz)-(7+i)| < \varepsilon$, i.e., $|f(z) - w| < \varepsilon$.

Example. Prove that $\lim_{z\to 0} z/\overline{z}$ does not exist.

Solution: Let $z = x + yi$ and $f(z) = z/\overline{z}$. The limit along the real axis (i.e., $y = 0$) applies to $f(z) = \frac{x}{x} = 1$, whereas the limit along the imaginary axis (i.e., $x = 0$) applies to $f(z) = \frac{yi}{-yi} = -1$. This shows that such limit cannot exist.

Definition. A complex function $f(z)$ can be expressed in rectangular form as $f(x+yi) = u(x, y) + iv(x, y)$, where u and v are both real functions in two variables. For example, if $f(z) = z^2$, then $f(x + yi) = x^2 + 2xyi - y^2$ and so $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$.

Theorem 3.1. Let $f(z) = u(x, y) + iv(x, y)$. Let $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$. Then

$$\lim_{z\to z_0} f(z) = w_0 \quad\Longleftrightarrow\quad \lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \quad\text{and}\quad \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0$$

Theorem 3.2. Let $\lim_{z\to z_0} f(z) = w_0$ and $\lim_{z\to z_0} g(z) = w_1$. Then

$$\lim_{z\to z_0} [f(z) + g(z)] = w_0 + w_1 \quad\text{and}\quad \lim_{z\to z_0} [f(z)g(z)] = w_0w_1 \quad\text{and}\quad \lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{w_0}{w_1}$$

Example. Note that $\lim_{z\to z_0} c = c$ and $\lim_{z\to z_0} z = z_0$. Thus $\lim_{z\to z_0} z^n = z_0^n$ and for any polynomial $P(z) = a_0+a_1z+a_2z^2+\cdots+a_nz^n$ with $a_i \in \mathbb{C}$, we have $\lim_{z\to z_0} P(z) = P(z_0)$.

Definition. A *δ-neighborhood* of $z_0 \in \mathbb{C}$ is the open disc $|z - z_0| < \delta$ for some $\delta > 0$. A *deleted neighborhood* means the neighborhood minus the center, i.e., $0 < |z - z_0| < \delta$. By the *neighborhood of infinity* (∞) we mean the region $|z| > \frac{1}{\delta}$ where $\delta > 0$ is arbitrarily small.

Theorem 3.3. The limits involving the neighborhood of infinity are given below.

$$\begin{aligned}
\lim_{z\to z_0} f(z) = \infty &\quad\Longleftrightarrow\quad \lim_{z\to z_0} \tfrac{1}{f(z)} = 0\\
\lim_{z\to \infty} f(z) = w &\quad\Longleftrightarrow\quad \lim_{z\to 0} f\left(\tfrac{1}{z}\right) = w\\
\lim_{z\to \infty} f(z) = \infty &\quad\Longleftrightarrow\quad \lim_{z\to 0} \tfrac{1}{f\left(\frac{1}{z}\right)} = 0
\end{aligned}$$

Example. Prove that (a) $\lim_{z\to -1} \frac{iz+3}{z+1} = \infty$ (b) $\lim_{z\to\infty} \frac{2z+i}{z+1} = 2$ (c) $\lim_{z\to\infty} \frac{2z^3-1}{z^2+1} = \infty$.

Solutions: We show $\lim_{z\to z_0} \frac{1}{f(z)} = 0$ for (a): $\lim_{z\to -1} \frac{z+1}{iz+3} = \frac{-1+1}{i(-1)+3} = \frac{0}{-1+3} = 0$.

Show $\lim_{z\to 0} f(\frac{1}{z}) = w$ for (b): $\lim_{z\to 0} \frac{2(\frac{1}{z})+i}{(\frac{1}{z})+1} = \frac{2+iz}{1+z} = \frac{2+i(0)}{1+0} = \frac{2}{1} = 2$.

Show $\lim_{z\to 0} \frac{1}{f(\frac{1}{z})} = 0$ for (c): $\lim_{z\to 0} \frac{1}{\frac{2(\frac{1}{z})^3-1}{(\frac{1}{z})^2+1}} = \frac{(\frac{1}{z})^2+1}{2(\frac{1}{z})^3-1} = \frac{z+z^3}{2-z^3} = \frac{0+0^3}{2-0^3} = \frac{0}{2} = 0$.

Definition. A complex function $f(z)$ is *continuous* at a point z_0 if both exist and

$$\lim_{z\to z_0} f(z) = f(z_0)$$

Example. By Theorem 3.2, if $f(z)$ and $g(z)$ are both continuous at z_0, then so are the functions $f+g$, fg, and f/g when $g(z_0)\neq 0$. Moreover, the composition $g\circ f$ is also continuous at z_0, provided that g is continuous at $f(z_0)$.

Theorem 3.4. Let $f(z)=u(x,y)+iv(x,y)$ and $z_0=x_0+iy_0$. Then f is continuous at z_0 if and only if u and v are both continuous at (x_0,y_0).

Theorem 3.5. If $f(z)$ is continuous at z_0 and $f(z_0)\neq 0$, then there exists a neighborhood D of z_0 such that $f(z)\neq 0$ for all $z\in D$.

Theorem 3.6. If $f(z)$ is continuous in a closed and bounded region R, then there exists $M\in\mathbb{R}$ such that $|f(z)|\leq M$ for all $z\in R$, and there exists $z_0\in R$ such that $|f(z_0)|=M$.

Proof. Let $f(z)=u(x,y)+iv(x,y)$. Then $|f(z)|=\sqrt{u(x,y)^2+v(x,y)^2}$, a continuous real function in two variables, so this theorem follows as a result in Calculus. ▽

Derivatives

Definition. The complex function $f(z)$ is *differentiable* at $z_0\in\mathbb{C}$ if f is defined on a neighborhood of z_0 such that the following *derivative* of f at z_0 exists:

$$f'(z_0):=\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}=\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}=\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z}=\frac{dw}{dz}$$

Example. Find the derivative of $f(z)=z^2$ where it is differentiable.

Solution: $f'(z)=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}=\lim_{\Delta z\to 0}\frac{(z+\Delta z)^2-z^2}{\Delta z}=\lim\frac{z^2+2z\Delta z+(\Delta z)^2-z^2}{\Delta z}=\lim_{\Delta z\to 0}2z+\Delta z=2z$. Hence, $f(z)=z^2$ is differentiable for all $z\in\mathbb{C}$, and $f'(z)=2z$.

Example. Show that $f(z)=\overline{z}$ is not differentiable anywhere.

Solution: $f'(z)=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}=\lim_{\Delta z\to 0}\frac{\overline{z+\Delta z}-\overline{z}}{\Delta z}$. If $\Delta z\to 0$ horizontally, i.e., $\Delta z=\Delta x+0i$, then we get $\lim_{\Delta z\to 0}\frac{\overline{z}+\Delta x-\overline{z}}{\Delta x}=1$. But if $\Delta z\to 0$ vertically, i.e., $\Delta z=0+i\Delta y$ then we have $\lim_{\Delta z\to 0}\frac{\overline{z}-i\Delta y-\overline{z}}{i\Delta y}=-1$. This shows that $f'(z)$ does not exist for any value of $z\in\mathbb{C}$.

Example. Find the domain where $f(z)=|z|^2$ is differentiable.

Solution: Note that $|z|^2=z\overline{z}$, so $f'(z)=\lim_{\Delta z\to 0}\frac{(z+\Delta z)\overline{(z+\Delta z)}-(z\overline{z})}{\Delta z}=\overline{z}+\overline{\Delta z}+\frac{z\overline{\Delta z}}{\Delta z}$. As in the preceding example, $\lim_{\Delta z\to 0}\overline{\Delta z}/\Delta z$ does not exist, hence neither does $f'(z)$, unless $z=0$. This is the only point at which f is differentiable, where $f'(0)=0$.

Theorem 3.7. Some familiar derivative formulas hold:

(a) $\frac{d}{dz}c=0\ (c\in\mathbb{C})$ (b) $\frac{d}{dz}z=1$ (c) $\frac{d}{dz}[cf(z)]=cf'(z)$ (d) $\frac{d}{dz}z^n=nz^{n-1}\ (n\in\mathbb{N})$

as well as the sum, product, quotient, and chain rules:

$$[f(z)+g(z)]'=f'(z)+g'(z) \qquad [f(z)g(z)]'=f'(z)g(z)+f(z)g'(z)$$

$$[g(f(z))]'=g'(f(z))\,f'(z) \qquad \left[\frac{f(z)}{g(z)}\right]'=\frac{f'(z)g(z)-f(z)g'(z)}{g(z)^2}$$

Example. Evaluate the derivative of (a) $f(z)=(2z^2+i)^5$ (b) $f(z)=(2z^2+i)^5/(z-1)$.

Solution: (a) $f'(z)=5(2z^2+i)^4(4z)=20z(2z^2+i)^4$ and

(b) $f'(z)=\frac{20z(2z^2+i)^4\,(z-1)-(2z^2+i)^5}{(z-1)^2}=\frac{(2z^2+i)^4\,\{20z^2-20z-(2z^2+i)\}}{(z-1)^2}=\frac{(2z^2+i)^4\,(18z^2-20z-i)}{(z-1)^2}$.

4 Analytic Functions

Definition. A complex function $f(z)$ is *analytic* at z_0 if $f'(z)$ exists for all z in some neighborhood of z_0. We call $f(z)$ *entire* if it is analytic for all $z \in \mathbb{C}$. If $f'(z)$ exists in some deleted neighborhood of z_0 but not at z_0 itself, then z_0 is called a *singular point* or *singularity* of $f(z)$.

Example. We illustrate with some familiar functions.
(a) Every polynomial is entire.
(b) The function $f(z) = \frac{1}{z}$ is analytic for all $z \neq 0$, so $z_0 = 0$ is the only singular point.
(c) The function $f(z) = |z|^2$ has derivative only at $z = 0$, so it is not analytic anywhere and has no singular point.
(d) Any rational function f/g has a singular point z_0 if and only if $g(z_0) = 0$.

Theorem 4.1. If $f(z)$ is analytic in a domain D and $g(z)$ is analytic in a domain containing the range of f, then $g \circ f$ is analytic in D.

Theorem 4.2 (Cauchy-Riemann)**.** Suppose that $f(z) = u(x, y) + iv(x, y)$ and that $f'(z)$ exists at $z_0 = x_0 + iy_0$. Then the partial derivatives of u and v must exist at (x_0, y_0) and

$$u_x(x_0, y_0) = v_y(x_0, y_0) \quad \text{and} \quad u_y(x_0, y_0) = -v_x(x_0, y_0)$$

Moreover, the derivative at z_0 is given by $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$.

Definition. The pair of identities $u_x = v_y$ and $u_y = -v_x$ is called the *Cauchy-Riemann equations*, a necessary condition for the existence of $f'(z)$ where $f(z)$ is differentiable.

Proof. Let $\Delta z = \Delta x + i\Delta y$ so that we may write

$$\begin{aligned}\Delta w =& f(z_0 + \Delta z) - f(z_0) \\ =& [u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)] + i\,[v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)]\end{aligned}$$

If $f'(z_0) = \lim_{\Delta z \to 0} \Delta w/\Delta z$ exists, then the same limit exists as $(\Delta x, \Delta y) \to (0, 0)$ horizontally along the line $(\Delta x, 0)$, in which case

$$\begin{aligned} f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} &= \lim_{\Delta x \to 0} \left(\frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i\, \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \right) \\ &= u_x(x_0, y_0) + iv_x(x_0, y_0) \end{aligned}$$

Similarly if $(\Delta x, \Delta y) \to (0, 0)$ vertically along the line $(0, \Delta y)$, then we have

$$\begin{aligned} f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} &= \lim_{\Delta y \to 0} \left(\frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + i\, \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i\Delta y} \right) \\ &= v_y(x_0, y_0) - iu_y(x_0, y_0) \end{aligned}$$

Matching the two yields the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$. ▽

Example. Let $f(z) = \overline{z} = x - yi$. We have $u(x, y) = x \to u_x = 1;\ u_y = 0$ and $v(x, y) = -y \to v_x = 0;\ v_y = -1$. Since the Cauchy-Riemann equation $u_x = v_y$ is false for all $z \in \mathbb{C}$, we conclude that $f(z) = \overline{z}$ is not differentiable anywhere.

Theorem 4.3 (Cauchy-Riemann)**.** Suppose that $f(z) = u(x, y) + iv(x, y)$ is defined on a neighborhood R of $z_0 = x_0 + iy_0$. If the partial derivatives of u and v (1) exist for all $(x, y) \in R$ and (2) continuous at (x_0, y_0) and (3) satisfy the Cauchy-Riemann equations, then $f'(z_0)$ exists.

Example. Find the domain where $f(z) = z^2$ is analytic and find its derivative.

Solution: Since $(x+yi)^2 = x^2+2xyi-y^2$, then $u(x,y) = x^2-y^2 \to u_x = 2x;\ u_y = -2y$ and $v(x,y) = 2xy \to v_x = 2y;\ v_y = 2x$. The two Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ hold for all (x,y) and all four are continuous functions. So $f(z) = z^2$ is entire (analytic for all $z \in \mathbb{C}$) where $f'(z) = u_x + iv_x = 2x + i2y = 2(x+yi) = 2z$.

Example. Find the domain where $f(z) = |z|^2$ is analytic and find its derivative.

Solution: Here $u(x,y) = x^2 + y^2$ and $v(x,y) = 0$. To have $u_x = v_y$ and $u_y = -v_x$ is possible only for $(x,y) = (0,0)$. Since all four are continuous functions, we see that $f(z) = |z|^2$ has derivative only at $z = 0$ (hence, f is not analytic anywhere), where $f'(0) = u_x(0) + iv_x(0) = 0$.

Example. Find the domain where $f(z) = e^z$ is analytic and find its derivative.

Solution: We have $u(x,y) = e^x \cos y \to u_x = e^x \cos y;\ u_y = -e^x \sin y$ and $v(x,y) = e^x \sin y \to v_x = e^x \sin y;\ v_y = e^x \cos y$. The conditions of Cauchy-Riemann theorem are satisfied for all $z \in \mathbb{C}$, hence $f(z) = e^z$ is an entire function with $f'(z) = u_x + iv_x = e^x \cos y + ie^x \sin y = e^z$.

Theorem 4.4. Since $\frac{d}{dz} e^z = e^z$ for all $z \in \mathbb{C}$, we can show that also for all $z \in \mathbb{C}$,

(1) $\frac{d}{dz} \cos z = -\sin z$ (2) $\frac{d}{dz} \sin z = \cos z$ (3) $\frac{d}{dz} \cosh z = \sinh z$ (4) $\frac{d}{dz} \sinh z = \cosh z$

Theorem 4.5 (Cauchy-Riemann, polar form)**.** Let $f(z) = u(r,\theta) + iv(r,\theta)$ be defined on a neighborhood R of $z_0 = r_0 e^{i\theta_0}$. If the partial derivatives of u and v (1) exist for all $(r,\theta) \in R$ and (2) continuous at $z_0 = r_0 e^{i\theta_0}$ and (3) satisfy the Cauchy-Riemann equations in polar form, then $f'(z_0)$ exists. The polar Cauchy-Riemann equations are

$$ru_r(r_0,\theta_0) = v_\theta(r_0,\theta_0) \quad \text{and} \quad u_\theta(r_0,\theta_0) = -rv_r(r_0,\theta_0)$$

Both are necessary conditions for the existence of $f'(z)$, which is given in general by

$$f'(z) = e^{-i\theta}(u_r + iv_r)$$

Example. Show that $f(z) = z^2$ is entire with $f'(z) = 2z$.

Solution: $f(re^{i\theta}) = r^2 e^{i2\theta}$ so $u(r,\theta) = r^2 \cos(2\theta) \to u_r = 2r\cos(2\theta);\ u_\theta = -2r^2 \sin(2\theta)$ and $v(r,\theta) = r^2 \sin(2\theta) \to v_r = 2r\sin(2\theta);\ v_\theta = 2r^2\cos(2\theta)$. The C-R equations $ru_r = v_\theta$ and $u_\theta = -rv_r$ hold for all $z \in \mathbb{C}$, hence f is entire. And $f'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}(2r\cos 2\theta + i2r\sin 2\theta) = e^{-i\theta}(2r)(\cos 2\theta + i\sin 2\theta) = e^{-i\theta}(2r)(e^{i2\theta}) = 2re^{i\theta} = 2z$.

Theorem 4.6. We have $\frac{d}{dz} \operatorname{Log} z = \frac{1}{z}$ in the domain $\mathbb{C}^* - (-\infty, 0]$. In general we also have $\frac{d}{dz} \log z = \frac{1}{z}$ in the domain $\mathbb{C}^* - \{z \in \mathbb{C} \mid \arg z = \theta_0\}$, if we have selected the single-valued function $\log z$ with restriction $\theta_0 < \arg z < \theta_0 + 2\pi$.

Proof. Write $\operatorname{Log} z = \ln r + i\theta$, which we know is continuous but only in the indicated domain. We have $u = \ln r$ and $v = \theta$, hence $u_r = \frac{1}{r};\ v_r = 0$ and $u_\theta = 0;\ v_\theta = 1$. The C-R equations hold: $ru_r = v_\theta$ and $u_\theta = -rv_r$ for all z in the domain, where $f'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}(\frac{1}{r} + 0) = \frac{1}{r}e^{-i\theta} = \frac{1}{z}$. ▽

Example. Show that the derivative of $f(z) = \frac{1}{z}$ exists for all $z \neq 0$ with $f'(z) = -1/z^2$.

Solution: $f(re^{i\theta}) = \frac{1}{r}e^{-i\theta} \to u(r,\theta) = \frac{1}{r}\cos(-\theta) = \frac{1}{r}\cos\theta$ and $v(r,\theta) = \frac{1}{r}\sin(-\theta) = -\frac{1}{r}\sin\theta$. So $u_r = -\frac{1}{r^2}\cos\theta;\ u_\theta = -\frac{1}{r}\sin\theta$ and $v_r = \frac{1}{r^2}\sin\theta;\ v_\theta = -\frac{1}{r}\cos\theta$. The C-R equations $ru_r = v_\theta$ and $u_\theta = -rv_r$ hold for all $r > 0$, so $f(z) = \frac{1}{z}$ is analytic for all $z \neq 0$. And $f'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}(-\frac{1}{r^2}\cos\theta + i\frac{1}{r^2}\sin\theta) = e^{-i\theta}(\frac{1}{r^2})(-\cos\theta + i\sin\theta) = e^{-i\theta}(\frac{1}{r^2})(-\cos(-\theta) - i\sin(-\theta)) = e^{-i\theta}(\frac{1}{r^2})(-e^{-i\theta}) = -\frac{1}{r^2}e^{-i2\theta} = -\frac{1}{r^2 e^{i2\theta}} = -\frac{1}{z^2}$.

Example. Show that $f(z) = z^{\frac{1}{3}}$ is analytic for all $z \in \mathbb{C} - [0, \infty)$ with $f'(z) = \frac{1}{3z^{2/3}}$.

Theorem 4.7. If $f'(z) = 0$ for all $z \in D$, then $f(z)$ is constant in D.

Proof. Let $f(z) = u + iv$ and so by Cauchy-Riemann theorem $f'(x + yi) = u_x + iv_x$. Now $f' = 0$ implies that $u_x = 0$ and $v_x = 0$, and therefore also $v_y = u_x = 0$ and $u_y = -v_x = 0$. Since $u_x = u_y = 0$ in D, we have there $u(x, y) = c$, a real constant, and similarly $v(x, y) = d$, giving us $f(z) = c + di$, a complex constant. ▽

Harmonic Functions

Definition. A real function $h(x, y)$ is *harmonic* in a domain D if it has continuous first and second partial derivatives such that $h_{xx}(x, y) + h_{yy}(x, y) = 0$ for all $(x, y) \in D$.

Theorem 4.8. Let $f(z) = u(x, y) + iv(x, y)$. If $f(z)$ is analytic for all z in a domain D, then both u and v are harmonic for all $(x, y) \in D$.

Example. We know that $e^z = e^x(\cos y + i \sin y)$ is entire. Here $u = e^x \sin y$ and $v = e^x \sin y$, hence $u_x = e^x \sin y \to u_{xx} = e^x \sin y$ and $u_y = e^x \cos y \to u_{yy} = -e^x \sin y$. It follows that $u_{xx} + u_{yy} = 0$, so $u(x, y)$ is harmonic for all (x, y), and similarly for $v(x, y)$.

Theorem 4.9. Let $f(z) = u(x, y) + iv(x, y)$. Then $f(z)$ is analytic in a domain D if and only if v is a *harmonic conjugate* of u, meaning that both u and v are harmonic and that they satisfy the Cauchy-Riemann equations in D.

Example. Let $u = x^2 - y^2$ and $v = 2xy$. Show that v is a harmonic conjugate of u for all (x, y). Is u also a harmonic conjugate of v?

Example. Show that $u(x, y) = y^3 - 3x^2y$ is harmonic, then find a harmonic conjugate $v(x, y)$ such that $f(z) = u + iv$ is entire.

Solution: $u_x = -6xy \to u_{xx} = -6y$ and $u_y = 3y^2 - 3x^2 \to u_{yy} = 6y$, so $u_{xx} + u_{yy} = 0$. Now $u_x = v_y \to -6xy = v_y \to v = \int -6xydy \to v = -3xy^2 + g(x)$. And $u_y = -v_x \to 3y^2 - 3x^2 = -(-3y^2 + g'(x)) \to 3x^2 = g'(x) \to g(x) = x^3 + C$, a constant. Thus $v(x, y) = -3xy^2 + x^3 + C$. Note that for $C = 0$, we have $f(z) = u + iv = y^3 - 3x^2y - i3xy^2 + ix^3 = iz^3$.

Example. Find a harmonic conjugate of $u = e^x \cos y$ and show how this gives $f(z) = e^z$.

5 Contour Integrals

Definition. Let $w : \mathbb{R} \to \mathbb{C}$ be of the form $w(t) = u(t) + iv(t)$ for some real functions (in one variable) u and v. If the integrals of u and v exist over the real interval $[a, b]$, then we define

$$\int_a^b w(t)\, dt := \int_a^b u(t)\, dt + i \int_a^b v(t)\, dt$$

Example. Evaluate $\int_0^1 (1 + it)^2\, dt$.

Solution: $\int_0^1 (1 - t^2) + i(2t)\, dt = \int_0^1 (1 - t^2)dt + i \int_0^1 2t\, dt = (t - \frac{1}{3}t^3)_0^1 + i(t^2)_0^1 = (1 - \frac{1}{3}) + i(1) = \frac{2}{3} + i$.

Theorem 5.1. It is clear that if we have $W(t) = U(t) + iV(t)$ such that $\frac{dW}{dt} = w(t)$, then $\frac{dU}{dt} = u(t)$ and $\frac{dV}{dt} = v(t)$. In this case therefore,

$$\int_a^b w(t)\, dt = W(b) - W(a)$$

Example. Evaluate $\int_0^{\pi/4} e^{it}\, dt$ in two ways.

Solution: (1)By definition, $\int(\cos t + i\sin t)dt = \int \cos t dt + i\int \sin t dt = (\sin t)_0^{\pi/4} + i(-\cos t)_0^{\pi/4} = \frac{1}{\sqrt{2}} - i(\frac{1}{\sqrt{2}} - 1)$.

(2) Using anti-derivative, $\int e^{it}dt = (\frac{1}{i}e^{it})_0^{\pi/4} = -i(e^{i\pi/4} - e^0) = -i(\cos(\pi/4) + i\sin(\pi/4) - 1) = -i(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} - 1) = \frac{1}{\sqrt{2}} - i(\frac{1}{\sqrt{2}} - 1)$.

Theorem 5.2. Let $w(t) : \mathbb{R} \to \mathbb{C}$ be piecewise continuous on the real interval $a \le t \le b$. Then

$$\left| \int_a^b w(t)\, dt \right| \le \int_a^b |w(t)|\, dt$$

Proof. We write $\int_a^b w(t)\, dt = re^{i\theta}$, assuming that $\left|\int_a^b w(t)\, dt\right| = r$. Since θ is constant, we have $r = \int_a^b e^{-i\theta} w(t)\, dt$ and, since $r \in \mathbb{R}$, also that $r = \int_a^b \text{Re}[e^{-i\theta}w(t)]\, dt$. Now

$$\text{Re}[e^{-i\theta}w(t)] \le |e^{-i\theta}w(t)| = |e^{-i\theta}|\, |w(t)| = |w(t)|$$

which gives us $\left|\int_a^b w(t)\, dt\right| = r = \int_a^b \text{Re}[e^{-i\theta}w(t)]\, dt \le \int_a^b |w(t)|\, dt$. ▽

Definition. A *curve* in the complex plane is given by the set $C = \{z(t) = x(t) + iy(t) \mid a \le t \le b\}$ where $x(t), y(t)$ are continuous real functions in t. A curve C is *simple* if $z(t)$ is one-to-one within the interval $t \in [a, b]$. We say C is *closed* when $z(a) = z(b)$, and *simple closed* if it is closed but one-to-one on (a, b).

Example. Some examples of a complex xurve:
(a) $z(t) = e^{it}$ $(0 \le t \le 2\pi)$ is a simple closed curve from 1 to 1 in the positive (counter-clockwise) direction.
(b) $z(x) = x + ix$ $(0 \le x \le 1)$ followed by $z(x) = x + i$ $(1 \le x \le 2)$ is a simple curve from 0 to $2 + i$.
(c) $z(\theta) = e^{-i\theta}$ $(0 \le \theta \le 2\pi)$ is a simple closed curve from 1 to 1, negative orientation.
(d) $z(\theta) = e^{i2\theta}$ $(0 \le \theta \le 2\pi)$ is a closed curve from 1 to 1, but not simple.
(e) $z(t) = z_0 + Re^{it}$ $(0 \le t \le 2\pi)$ is a simple closed curve, i.e., a circle with center at z_0 and radius R.

Definition. The curve C is *smooth* if $z'(t)$ exists and continuous on $[a, b]$ and $z'(t) \ne 0$ for all $t \in (a, b)$. C is called a *contour* if it is piecewise smooth curve.

Definition. Let C be a contour from $z_1 = z(a)$ to $z_2 = z(b)$. We define the *contour integral*

$$\int_C f(z)\, dz := \int_{z_1}^{z_2} f(z)\, dz := \int_a^b f\left(z(t)\right)\, z'(t)\, dt$$

Example. Evaluate $\int_C \overline{z}\, dz$ where C is the semicircle $z = 2e^{it}$ $(-\frac{\pi}{2} \le t \le \frac{\pi}{2})$.

Solution: Here $z(t) = 2e^{it} \to z'(t) = 2ie^{it}$, so $\int_C f(z)dz = \int_{-\pi/2}^{\pi/2}(2e^{-it})(2ie^{it})dt = 4i\int dt = 4\pi i$. Note the fact that $z = re^{i\theta} \to \overline{z} = re^{-i\theta}$.

Example. Evaluate $\int_C (-x + y - i3x^2)\, dz$ where (a) C_1 is the line from 0 to i (b) C_2 is the line from i to $1 + i$ (c) C_3 is the line from 0 to $1 + i$.

Solution: Here $x(t) = 0$ and $y(t) = t$ $(0 \le t \le 1) \to z(t) = it$ and $z'(t) = i$. So $\int_{C_1} f(z)dz = \int_0^1 (0 + t - 0)(i)dt = \frac{1}{2}t^2)_0^1(i) = \frac{1}{2}i$.

(b) Here $x(t) = t$ and $y(t) = 1$ $(0 \le t \le 1) \to z(t) = t + i$ and $z'(t) = 1$. So $\int_{C_2} f(z)dz = \int_0^1 (-t + 1 - i3t^2)(1)dt = -\frac{1}{2}t^2 + t - it^3)_0^1 = \frac{1}{2} - i$.

(c) Here $x(t) = t$ and $y(t) = t$ $(0 \le t \le 1) \to z(t) = t + it$ and $z'(t) = 1 + i$. So $\int_{C_3} f(z)dz = \int_0^1 (-t + t - i3t^2)(1 + i)dt = -it^3)_0^1(1 + i) = 1 - i$.

Note that even though $C_1 + C_2$ has the same endpoints as C_3, we have $\int_{C_1} f(z)dz + \int_{C_2} f(z)dz \neq \int_{C_3} f(z)dz$.

Example. Evaluate $\int_C z^2\, dz$ where C is the semicircle from 3 to -3 in positive direction.

Solution: Let $z(t) = 3e^{it}$ $(0 \le t \le \pi) \to z'(t) = 3ie^{it}$. So $\int_C z^2 dz = \int_0^\pi 9e^{2it}(3ie^{it})dt = 27i \int e^{3it}dt = 27i\frac{1}{3i}e^{3it})_0^\pi = 9(e^{3i\pi} - e^0) = 9(-1 - 1) = -18$.

Theorem 5.3. The following properties hold for contour integrals.

1. $\int_C z_0 f(z)\, dz = z_0 \int_C f(z)\, dz$ for any constant $z_0 \in \mathbb{C}$
2. $\int_C [f(z) + g(z)]\, dz = \int_C f(z)\, dz + \int_C g(z)\, dz$
3. $\int_{-C} f(z)\, dz = -\int_C f(z)\, dz$ where $-C$ is the contour C traversed in negative direction, i.e., $z_{-C}(t) = z_C(-t)$
4. $\int_C f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz$ if the contour C consists of the contour C_1 followed by C_2

Example. Evaluate $\int_{C_1} z^2\, dz$ where C_1 is the line from 0 to $1 + i$.

Solution: Here $x(t) = t$ and $y(t) = t$ $(0 \le t \le 1) \to z(t) = t + it$ and $z'(t) = 1 + i$. So $\int_{C_1} z^2 dz = \int_0^1 (t + it)^2(1 + i)dt = \int t^2(i + i)^3 dt = (1 + i)^3 \frac{1}{3}(t^3)_0^1 = \frac{1}{3}(1 + i)^3 = \frac{1}{3}(-2 + 2i) = \frac{2}{3}(-1 + i)$.

Example. Evaluate $\int_{C_2} z^2\, dz$ where C_2 is the curve from 0 to $1+i$ along parabola $y = x^2$.

Solution: Here $x(t) = t$ and $y(t) = t^2$ $(0 \le t \le 1) \to z(t) = t + it^2$ and $z'(t) = 1 + 2it$. So $\int_{C_2} z^2 dz = \int_0^1 (t + it^2)^2(1 + 2it)dt = ... = \int (t^2 - 5t^4) + i(4t^3 - 2t^5)dt = \frac{1}{3}(t^3 - t^5)_0^1 + i(t^4 - \frac{1}{3}t^6) - 0^1 = (\frac{1}{3} - 1) + i(1 - \frac{1}{3}) = \frac{2}{3}(-1 + i)$. Note that $\int_{C_1} z^2 dz = \int_{C_2} z^2 dz$ and C_1 and C_2 are two contours with the same end-points.

Example. Evaluate $\int_{C_3} z^{-2}\, dz$ where C_3 is the semicircle from 1 to -1, positive.

Solution: Here $z(t) = e^{it}$ $(0 \le t \le \pi) \to z'(t) = ie^{it}$. So $\int_{C_3} z^{-2} dz = \int_0^\pi e^{-2it} ie^{it} dt = i \int e^{-it} dt = \frac{i}{-i}(e^{-it})_0^\pi = -(e^{-i\pi} - e^0) = -(-1 - 1) = 2$.

Example. Evaluate $\int_{C_4} z^{-2}\, dz$ where C_4 is the semicircle from 1 to -1, negative.

Solution: Here $z(t) = e^{-it}$ $(0 \le t \le \pi) \to z'(t) = -ie^{-it}$. So $\int_{C_4} z^{-2} dz = \int_0^\pi e^{2it}(-i)e^{-it} dt = -i \int e^{it} dt = \frac{-i}{i}(e^{it})_0^\pi = -(e^{i\pi} - e^0) = -(-1 - 1) = 2$. Note that $\int_{C_3} z^{-2} dz = \int_{C_4} z^{-2} dz$ and C_3 and C_4 are two contours with the same endpoints.

Example. Evaluate $\int_{C_5} z^{-2}\, dz$ where C_5 is the unit circle, positive.

Solution: Here $z(t) = e^{it}$ $(0 \le t \le 2\pi) \to z'(t) = ie^{it}$. So $\int_{C_5} z^{-2} dz = \int_0^{2\pi} e^{-2it} ie^{it} dt = i \int e^{-it} dt = \frac{i}{-i}(e^{-it})_0^{2\pi} = -(e^{-2i\pi} - e^0) = -(1 - 1) = 0$.

Another solution: Note that $C_5 = C_3 - C_4$ with C_3 and C_4 from the previous examples, but C_4 with negative orientation. So $\int_{C_5} z^{-2} dz = \int_{C_3} z^{-2} dz - \int_{C_4} z^{-2} dz = 2 - 2 = 0$.

Example. Evaluate $\int_{C_6} \frac{1}{z}\, dz$ where C_6 is the semicircle from 1 to -1, positive.

Solution: Here $z(t) = e^{it}$ $(0 \le t \le \pi) \to z'(t) = ie^{it}$. So $\int_{C_6} \frac{1}{z} dz = \int_0^\pi e^{-it} ie^{it} dt = \int i dt = i(\pi - 0) = i\pi$.

Example. Evaluate $\int_{C_7} \frac{1}{z}\, dz$ where C_7 is the semicircle from 1 to -1, negative.

Solution: $z(t) = e^{-it}$ $(0 \leq t \leq \pi) \to z'(t) = -ie^{-it}$. So $\int_{C_7} \frac{1}{z} dz = \int_0^{\pi} e^{it}(-i)e^{-it} dt = \int (-i) dt = -i(\pi - 0) = -i\pi$. Note that $\int_{C_6} \frac{1}{z} dz \neq \int_{C_7} \frac{1}{z} dz$ even though C_6 and C_7 have the same endpoints. (Why?)

Example. Evaluate $\int_{C_8} \frac{1}{z}\, dz$ where C_8 is the unit circle, positive.

Solution: Here $z(t) = e^{it}$ $(0 \leq t \leq 2\pi) \to z'(t) = ie^{it}$. So $\int_{C_8} \frac{1}{z} dz = \int_0^{2\pi} e^{-it} ie^{it} dt = \int i dt = i(2\pi - 0) = 2\pi i$.

Another solution: Note that $C_8 = C_6 - C_7$ with C_6 and C_7 from before, but C_7 with negative orientation. So $\int_{C_8} \frac{1}{z} dz = \int_{C_6} \frac{1}{z} dz - \int_{C_7} \frac{1}{z} dz = i\pi - (-i\pi) = 2i\pi$. Note that in general $\int_C f(z) dz$ may not equal zero even if C is a closed contour.

Example. Show that $\int_C \frac{dz}{z} = 2\pi i$ for any circle C with center at 0 and radius R.

Example. Show that $\int_C \overline{z}\, dz = 4\pi i$ where C is the semicircle $z = 2e^{i\theta}$ $(-\pi/2 \leq \theta \leq \pi/2)$.

Example. Let C_1 be the line from 0 to i, let C_2 from i to $1+i$, let C_3 from 0 to $1+i$. Let $f(z) = -x + y - i3x^2$. Show (a) $\int_{C_1} f(z)\, dz = i/2$ (b) $\int_{C_2} f(z)\, dz = -i + 1/2$ (c) $\int_{C_3} f(z)\, dz = 1 - i$. Note that $\int_{C_1+C_2} f(z)\, dz \neq \int_{C_3} f(z)\, dz$ despite the same endpoints.

Example. Show that $\int_C z^{1/2}\, dz = -2\sqrt{3}(1+i)$ where C is the semicircle $z = 3e^{i\theta}$ $(0 \leq \theta \leq \pi)$, using the branch cut at positive x-axis.

Theorem 5.4. Let C be a contour of arc-length L. Suppose that $f(z)$ is piecewise continuous on C and that there exists a positive real number M such that $|f(z)| \leq M$ for all z in some domain containing C. Then

$$\left| \int_C f(z)\, dz \right| \leq ML$$

Proof. Let $z = z(t)$ be the parametric equation with $a \leq t \leq b$. Observe that on C,

$$|\, f(z(t))\; z'(t)\, | = |f(z(t))|\;\; |z'(t)| \leq M\; |z'(t)|$$

so then by Theorem 5.2

$$\left| \int_C f(z)\, dz \right| = \left| \int_a^b f(z(t))\, z'(t)\, dt \right| \leq \int_a^b |\, f(z(t))\, z'(t)\, |\; dt \leq M \int_a^b |z'(t)|\, dt$$

But $\int_a^b |z'(t)|\, dt = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt$ is just the arc-length of the contour C. ▽

Integrals as Anti-Derivatives

Theorem 5.5. The following are equivalent, where $f(z)$ is continuous on D.

1. $f(z)$ has an anti-derivative $F(z)$ such that $F'(z) = f(z)$ for all $z \in D$.
2. $\int_C f(z)\, dz = F(z_2) - F(z_1)$ for all contour $C \subseteq D$ from z_1 to z_2.
3. $\int_C f(z)\, dz = 0$ for all closed contour $C \subseteq D$.

Example. Evaluate $\int_C z^2\, dz$ for any contour C from $z = 0$ to $z = 1+i$.

Solution: An anti-derivative $F(z) = \frac{1}{3}z^3$ is entire, so regardless the path taken, $\int_C z^2\, dz = \frac{1}{3}[z^3]_0^{1+i} = \frac{1}{3}(1+i)^3 = \frac{2}{3}(-1+i)$.

Example. Evaluate $\int_{|z|=R} \frac{1}{z^2}\, dz$.

Solution: An anti-derivative exists, $F(z) = -\frac{1}{z}$ with domain $\mathbb{C}^*$, which contains C, and C is a closed contour. Thus $\int_C \frac{1}{z^2}\, dz = 0$.

Example. Evaluate $\int_C (z+1)^3\, dz$ where C is the straight line from 0 to 1, followed by a straight line from 1 to $1+i$, then followed by a straight line from $1+i$ to i.

Solution: By anti-derivative, $\int_C z^{-2}dz = \frac{1}{4}[(z+1)^4]_0^i = \frac{1}{4}[(1+i)^4 - 1] = \frac{1}{4}(-4-1) = -\frac{5}{4}$.

Example. Evaluate $\int_{C_1} \frac{1}{z}\, dz$ where C_1 is the semicircle from $-i$ to i, positive.

Sol: We have solved this earlier, but this time we may use anti-derivative. (Why?) $\int_{C_1} \frac{1}{z} dz = (\operatorname{Log} z)_{-i}^{i} = \operatorname{Log} i - \operatorname{Log}(-i) = \ln 1 + i(\frac{\pi}{2}) - (\ln 1 + i(-\frac{\pi}{2})) = i\pi$.

Example. Evaluate $\int_{|z|=1} \frac{dz}{z}$.

Solution: Unfortunately, we cannot use the anti-derivative $\operatorname{Log} z$. (Why?) Since we have computed $\int_{C_1} \frac{1}{z}\, dz = i\pi$, where C_1 is the positive semicircle from $-i$ to i, we need just add with $\int_{C_2} \frac{1}{z}\, dz$, where C_2 is the positive semicircle from i to $-i$. We will use a branch of $\log z$ with $0 < \theta < 2\pi$: (Why? Or just use parametric equation.) $\int_{C_2} \frac{1}{z} dz = (\log z)_i^{-i} = \log(-i) - \log(i) = \ln 1 + i(\frac{3\pi}{2}) - (\ln 1 + i(\frac{\pi}{2})) = i\pi$. Hence, $\int_{|z|=1} \frac{dz}{z} = i\pi + i\pi = 2\pi i$.

Example. Evaluate $\int_C \cos z\, dz$ where C is the straight line from 0 to $2i$.

Solution: $\int_C \cos z dz = (\sin z)_0^{2i} = \sin 2i - \sin 0 = \sin 2i = \frac{e^{-2} - e^{2}}{2i} = \frac{e^2 - e^{-2}}{2} i$.

Example. Evaluate $\int_C \overline{z}\, dz$ where C is the straight line from i to 1.

Solution: Anti-derivative unknown, so we must use definition with parametric equation: $z(t) = t + (1-t)i$ $(0 \le t \le 1) \to z'(t) = 1 - i$. So $\int_C \overline{z} dz = \int_0^1 (t - (1-t)i)(1-i)dt = (1-i)(\frac{1}{2}t^2 + \frac{i}{2}(1-t)^2)_0^1 = (1-i)(\frac{1}{2} - \frac{i}{2}) = \frac{-2i}{2} = -i$.

6 Cauchy Integral Formulas

Theorem 6.1 (Cauchy-Goursat)**.** Let C be a simple closed contour such that $f(z)$ is analytic on C as well as inside C. Then

$$\int_C f(z)\, dz = 0$$

Proof. Let C be represented by $z(t) = x(t) + iy(t)$ with the interval $(a \le t \le b)$, and write $f(z) = u(x,y) + iv(x,y)$. Then

$$\begin{aligned}\int_C f(z)\, dz &= \int_a^b \big[u(x(t), y(t)) + iv(x(t), y(t))\big] \cdot \big[x'(t) + iy'(t)\big]\, dt \\ &= \int_a^b (udx - vdy) + i(udy + vdx)\end{aligned}$$

At this we call for Green's theorem from Calculus 4, i.e.,

$$\int_C Pdx + Qdy = \int\int_R (Q_x - P_y)\, dA$$

to resume and conclude that

$$\int_C f(z)\, dz = \int\int_R (-v_x - u_y)\, dA + i \int\int_R (u_x - v_y)\, dA$$

Both integrals equal zero according to the Cauchy-Riemann equations. ▽

Example. In particular, Cauchy-Goursat theorem applies when $f(z)$ is entire. Hence, by Theorem 5.5, an entire function always has an anti-derivative for all $z \in \mathbb{C}$.

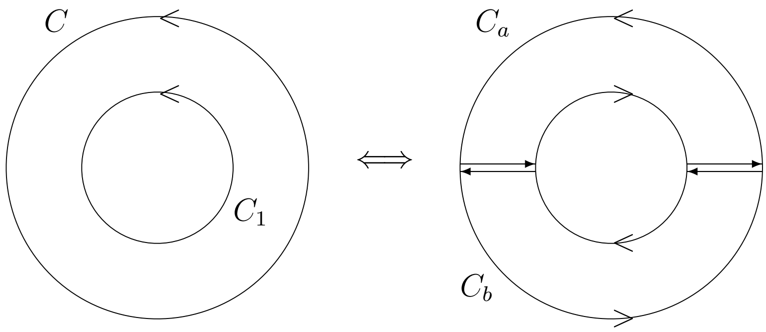
Example. We illustrate with some examples:
(1) By Cauchy-Goursat theorem, $\int_C e^{2z}\,dz = 0$ for any closed contour C because e^{2z} is an entire function.
(2) Recall that $\int_{|z|=1} z^{-2}\,dz = 0$ because an anti-derivative $F(z) = -\frac{1}{z}$ exists with domain $\mathbb{C}^*$ containing C. However, this result does not follow from Cauchy-Goursat theorem, since $f(z) = z^{-2}$ is not analytic at $z_0 = 0$, which is located inside C.
(3) Recall that $\int_{|z|=1} \frac{1}{z}\,dz = 2\pi i \neq 0$ although an anti-derivative $F(z) = \operatorname{Log} z$ exists because its domain $\mathbb{C}-(-\infty, 0]$ does not contain C. Neither does Cauchy-Goursat theorem apply here, since $f(z) = \frac{1}{z}$ is not analytic at $z_0 = 0$ which is located inside C.
(4) By Cauchy-Goursat theorem, $\int_{|z|=1} \frac{z^2+2}{z+2i}\,dz = 0$ because the integrand is analytic except for a unique singular point at $z_0 = -2i$, which is outside of C.

Theorem 6.2. Let C_1 be a closed contour inside a closed contour C such that $f(z)$ is analytic in the domain bounded between C_1 and C. Then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz$$

In general if $C_1, C_2, \ldots, C_n$ are closed and disjoint and all lie inside C, then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots + \int_{C_n} f(z)\,dz$$



Proof. The idea is to replace $C - C_1$ with two closed contours $C_a + C_b$ so that

$$\int_C f(z)\,dz - \int_{C_1} f(z)\,dz = \int_C f(z)\,dz + \int_{-C_1} f(z)\,dz =$$
$$\int_{C-C_1} f(z)\,dz = \int_{C_a} f(z)\,dz + \int_{C_b} f(z)\,dz = 0$$

by Cauchy-Goursat theorem since f is analytic inside C_a as well as inside C_b. ▽

Theorem 6.3 (Cauchy Integral Formula)**.** Let C be a closed contour such that $f(z)$ is analytic on and inside C. Then for all z_0 inside C we have

$$\int_C \frac{f(z)}{z - z_0}\,dz = 2\pi i\, f(z_0)$$

Proof. By Theorem 6.2, we replace C by the circle $|z - z_0| = R$, where the radius $R > 0$ can be arbitrarily small and to be determined later. Here $z(t) = z_0 + Re^{it}$ and $z'(t) = iRe^{it}$, so

$$\int_C \frac{dz}{z - z_0} = \int_0^{2\pi} \frac{iRe^{it}dt}{Re^{it}} = i\int_0^{2\pi} dt = 2\pi i$$

And since

$$\int_C \frac{f(z) - f(z_0)}{z - z_0}\, dz = \int_C \frac{f(z)}{z - z_0}\, dz - f(z_0)\int_C \frac{dz}{z - z_0} = \int_C \frac{f(z)}{z - z_0}\, dz - 2\pi i f(z_0)$$

it suffices to have the integral on the LHS equal zero, by proving that its modulus is less than any given $\varepsilon > 0$. To see this, as f is continuous at z_0, we can choose R so small that $|f(z) - f(z_0)| < \varepsilon/2\pi$ whenever $|z - z_0| \leq R$. In particular, this holds on C, where $|z - z_0| = R$. Hence,

$$\left|\int_C \frac{f(z) - f(z_0)}{z - z_0}\, dz\right| \leq \int_0^{2\pi} \left|\frac{f(z(t)) - f(z_0)}{R}\, z'(t)\right| dt < \int_0^{2\pi} \left|\frac{\varepsilon}{2\pi R}\, iRe^{it}\right| dt = \varepsilon$$

$\triangledown$

Example. Evaluate $\int_C \frac{z^2+1}{z^2-4}\, dz$ where C is a unit circle center at $z = 2$.

Solution: We have $z^2 - 4 = (z + 2)(z - 2)$, where the zero $z = 2$ is inside C, while $z = -2$ is outside C. Hence by Cauchy Integral formula,

$$\int_C \frac{z^2 + 1}{z^2 - 4}\, dz = \int_C \frac{\frac{z^2+1}{z+2}}{z - 2}\, dz = \int_C \frac{f(z)}{z - 2}\, dz = 2\pi i\, f(2) = 2\pi i \cdot \tfrac{5}{4} = \tfrac{5\pi}{2} i$$

Example. Evaluate $\int_C \frac{z+1}{z^2-4}\, dz$ where C is the circle with center at $z = 0$ and radius $R = 3$.

Solution: Let C_1 and C_2 be unit circles with center at $z = -2$ and $z = 2$, respectively. Cauchy Integral formula gives us

$$\int_C \frac{z + 1}{z^2 - 4}\, dz = \int_{C_1} \frac{z + 1}{z^2 - 4}\, dz + \int_{C_2} \frac{z + 1}{z^2 - 4}\, dz$$

For C_1 let $f(z) = \frac{z+1}{z-2}$ and let $z_0 = -2$:

$$\int_{C_1} \frac{(z + 1)/(z - 2)}{z + 2}\, dz = \int_{C_1} \frac{f(z)}{z - z_0}\, dz = 2\pi i\, f(-2) = 2\pi i \cdot \tfrac{-1}{-4} = \tfrac{\pi}{2} i$$

And for C_2 let $f(z) = \frac{z+1}{z+2}$ and let $z_0 = 2$:

$$\int_{C_2} \frac{(z + 1)/(z + 2)}{z - 2}\, dz = \int_{C_2} \frac{f(z)}{z - z_0}\, dz = 2\pi i\, f(2) = 2\pi i \cdot \tfrac{3}{4} = \tfrac{3\pi}{2} i$$

Thus $\int_C \frac{z+1}{z^2-4}\, dz = \frac{\pi}{2} i + \frac{3\pi}{2} = 2\pi i.$

Example. Show that $\int_{|z|=2} \frac{zdz}{(9-z^2)(z+i)} = \frac{\pi}{5}.$

Example. Show that $\int_{|z|=4} \frac{zdz}{z^2+9} = 2\pi i.$

Theorem 6.4 (Generalized Cauchy Integral Formula)**.** Let C be a closed contour such that $f(z)$ is analytic on and inside C. For all z_0 inside C, we have

$$\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n\,!}\,f^{(n)}(z_0)$$

where $f^{(n)}(z_0)$ stands for the n-th derivative of $f(z)$ evaluated at z_0.

Example. For example, $\int_{|z|=2} \frac{e^z}{z-1}\,dz = 2\pi i\,f(1) = 2e\pi i$, where $f(z) = e^z$. Another example, $\int_{|z|=1} \frac{1}{z}\,dz = 2\pi i\,f(0) = 2\pi i$, since $f(z) = 1$.

Proof. (Sketch) Write the Cauchy Integral Formula with s in place of z_0, then differentiate both sides repeatedly with respect to s:

$$\int_C \frac{f(z)}{z-s}\,dz = 2\pi i f(s) \to \int_C \frac{f(z)}{(z-s)^2}\,dz = 2\pi i f'(s) \to \int_C \frac{2f(z)}{(z-s)^3}\,dz = 2\pi i f''(s)$$
$$\to \int_C \frac{3\cdot 2f(z)}{(z-s)^4}\,dz = 2\pi i f^{(3)}(s) \to \cdots \to \int_C \frac{n!\,f(z)}{(z-s)^{n+1}}\,dz = 2\pi i f^{(n)}(s)$$

▽

Example. Evaluate $\int_C \frac{z^5-2z^3+1}{(z-1)^3}\,dz$ where C is the circle of radius 2 and center $z=0$.

Solution: Here $f(z) = z^5 - 2z^3 + 1$, $z_0 = 1$, and $n = 2$, so $\int_C \frac{z^5-2z^3+1}{(z-1)^3} dz = \frac{2\pi i}{2!}\cdot f''(1)$, where $f''(z) = 20z^3 - 12z \to f\,''(1) = 8 \to \int_C \frac{z^5-2z^3+1}{(z-1)^3} dz = 8\pi i$.

Example. Evaluate $\int_C \frac{1}{z^2}\,dz$ where C is the unit circle.

Solution: (We know the answer is zero because the anti-derivative $F(z) = -\frac{1}{z}$ exists in the domain $\mathbb{C}^*$ containing C, a closed contour, Theorem 5.5.) Here $f(z) = 1$, $z_0 = 0$, and $n = 1$, so $\int_C \frac{1}{z^2} dz = \frac{2\pi i}{1!}\cdot f'(0) = 0$ because $f'(z) = 0$.

Example. Show that $\int_{|z|=2} \frac{1}{(z^2+1)^2}\,dz = 0$.

Example. Show that $\int_{|z|=1} \frac{e^{2z}\,dz}{z^4} = \frac{8}{3}\pi i$.

Example. Show that $\int_{|z|=2} \frac{dz}{(z+i)^2} = 0$.

Applications to Real Integrals

Theorem 6.5. Let $P(x)$ and $Q(x)$ be real polynomials with $\deg Q - \deg P \geq 2$. The real improper integral

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = \lim_{R\to\infty} \int_C \frac{P(z)}{Q(z)}\,dz$$

where C is the closed contour consisting of the straight line from $-R$ to R on the real axis, followed by the positive semicircle from R to $-R$.

Proof. It suffices to show that the contour integral vanishes over the semicircle C' given by $z(t) = Re^{it}$ $(0 \leq t \leq \pi)$. Now the degree condition assures that the improper integral is convergent, hence on C', where $|z| = R$, we can find $M > 0$ such that $\left|\frac{P(z(t))}{Q(z(t))}\right| < \frac{M}{R^2}$. Then by Theorem 5.2,

$$\left|\int_{C'} \frac{P(z)}{Q(z)}\,dz\right| \leq \int_0^{\pi} \left|\frac{P(z(t))}{Q(z(t))}\,z'(t)\right|\,dt < \int_0^{\pi} \left|\frac{M}{R^2}\,Rie^{it}\right|\,dt = \frac{M\pi}{R}$$

and the limit as $R \to \infty$ is zero. ▽

Example. Evaluate $\int_{-\infty}^{\infty} \frac{dx}{x^2+1}$.

Solution: (Of course, we could use the anti-derivative $\arctan x$ here.) We evaluate $\int_C \frac{dz}{z^2+1}$ where C is the closed upper semi-circle of arbitrarily large radius. The denominator $z^2 + 1 = (z + i)(z - i)$, where $z = i$ is inside C and $z = -i$ outside. Hence, $\int_C \frac{dz}{z^2+1} = 2\pi i\, f(i)$ where $f(z) = \frac{1}{z+i} \to \int_{-\infty}^{\infty} \frac{dx}{x^2+1} = 2\pi i \frac{1}{i+i} = \pi$.

Example. Evaluate $\int_0^{\infty} \frac{x^2 dx}{(x^2+1)^2}$.

Solution: Note that $\int_0^{\infty} \frac{x^2 dx}{(x^2+1)^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+1)^2} = \frac{1}{2}\int_C \frac{z^2 dz}{(z^2+1)^2}$ where C is the closed large upper semi-circle as before. The denominator $(z^2+1)^2 = (z+i)^2(z-i)^2$, where only $z = i$ is inside C. So $\int_C \frac{z^2 dz}{(z^2+1)^2} = \frac{2\pi i}{1!} f'(i)$ where $f(z) = \frac{z^2}{(z+i)^2} \to f'(z) = \frac{2z(z+i)^2 - z^2 2(z+i)}{(z+i)^4}$ $\to f'(i) = \frac{2i(-4)-(-2)(2i)}{16} = \frac{-i}{4} \to \int_0^{\infty} \frac{x^2 dx}{(x^2+1)^2} = \frac{1}{2} \cdot 2\pi i \cdot \frac{-i}{4} = \frac{\pi}{4}$.

Example. Show that $\int_{-\infty}^{\infty} \frac{dx}{x^2-2x+2} = \pi$.

Theorem 6.6. A real integral of the form $\int_0^{2\pi} R(\cos x, \sin x)\, dx$, where $R(X, Y)$ is a rational function, can be transformed into a contour integral of the form $\int_C f(z)\, dz$, where C is the circle $z(x) = e^{ix}$ $(0 \le x \le 2\pi)$, via the substitutions

$$dx = \frac{dz}{iz}; \qquad \cos x = \frac{z + z^{-1}}{2}; \qquad \sin x = \frac{z - z^{-1}}{2i}$$

Proof. Let $z(t) = e^{it}$ and $z'(t) = ie^{it} = iz(t)$. Then $\frac{z(t)+z(t)^{-1}}{2} = \frac{e^{it}+e^{-it}}{2} = \cos t$ and similarly $\frac{z(t)-z(t)^{-1}}{2i} = \frac{e^{it}-e^{-it}}{2i} = \sin t$, and

$$\begin{aligned}\int_C R\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\frac{dz}{iz} &= \int_0^{2\pi} R\left(\frac{z(t)+z(t)^{-1}}{2}, \frac{z(t)-z(t)^{-1}}{2i}\right)\frac{z'(t)}{iz(t)}\, dt \\ &= \int_0^{2\pi} R(\cos t, \sin t)\; dt\end{aligned}$$

so the substitutions work. ▽

Example. Evaluate $\int_0^{2\pi} \frac{dx}{2+\cos x}$.

Solution: $\cos x = \frac{z+z^{-1}}{2} \to \frac{1}{2+\cos x} = \frac{1}{2+\frac{z+z^{-1}}{2}} = \frac{2}{4+z+z^{-1}} = \frac{2z}{4z+z^2+1}$ and so

$$\int_0^{2x} \frac{dx}{2+\cos x} = \frac{2}{i}\int_0^{2x} \frac{izdx}{4z+z^2+1} = \frac{2}{i}\int_C \frac{dz}{4z+z^2+1} = \frac{2}{i}\int_C f(z)\, dz$$

The denominator $4z + z^2 + 1 = 0 \to (z + 2)^2 - 3 = 0 \to z + 2 = \pm\sqrt{3} \to z = -2 \pm \sqrt{3}$, i.e., $\to 4z + z^2 + 1 = (z - (-2 + \sqrt{3}))(z + (-2 - \sqrt{3}))$. Only the first zero is inside C:

$$\int_C f(z)\, dz = \int_C \frac{\frac{1}{z-(-2-\sqrt{3})}}{z-(-2+\sqrt{3})} dz = 2\pi i\, f(-2+\sqrt{3}) = \frac{2\pi i}{(-2+\sqrt{3})-(-2-\sqrt{3})} = \frac{\pi i}{\sqrt{3}}$$

For the final answer, $\int_0^{2\pi} \frac{dx}{2+\cos x} = \frac{2}{i} \cdot \frac{\pi i}{\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$.

Example. Show that $\int_0^{2\pi} \frac{dx}{2+\sin x} = \frac{2\pi}{\sqrt{3}}$.

Theorem 6.6 can be generalized to integrands of the form $R(\cos nx, \sin nx)$, according to the substitutions

$$\cos nx = \frac{z^n + z^{-n}}{2} \quad \text{and} \quad \sin nx = \frac{z^n - z^{-n}}{2i}$$

Example. Evaluate $\int_{-\pi}^{\pi} \frac{\cos 2x\, dx}{5+4\cos x}$. (Can we replace $\int_{-\pi}^{\pi}$ by $\int_0^{2\pi}$?)

Solution: $\cos 2x = \frac{z^2+z^{-2}}{2} \to \frac{\cos 2x}{5+4\cos x} = \frac{\frac{z^2+z^{-2}}{2}}{5+4\frac{z+z^{-1}}{2}} = \frac{z^4+1}{10z^2+4z^3+4z}$ so the integral becomes

$$\int_0^{2x} \frac{\cos 2x dx}{5+4\cos x} = \frac{1}{i}\int_0^{2x} \frac{(z^4+1)iz dx}{10z^3+4z^4+4z^2} = \frac{1}{2i}\int_C \frac{(z^4+1)dz}{5z^3+2z^4+2z^2} = \frac{1}{2i}\int_C f(z)\, dz$$

The denominator $5z^3+2z^4+2z^2 = z^2(2z^2+5z+2) = z^2(2z+1)(z+2)$, with zeros $z = 0, -\frac{1}{2}$ inside C and $z = -2$ outside C. For $z = 0$ we have $\int_C f(z)dz = \frac{2\pi i}{1!} f'(0)$ where $f(z) = \frac{z^4+1}{2z^2+5z+2} \to f'(z) = \frac{(4z^3)(2z^2+5z+2)-(z^4+1)(4z+5)}{(2z^2+5z+2)^2} \to f'(0) = -\frac{5}{4}$. And for $z = -\frac{1}{2}$ we have $\int_C f(z)dz = 2\pi i\, f(-\frac{1}{2})$ where $f(z) = \frac{z^4+1}{z^2(2)(z+2)} \to f(-\frac{1}{2}) = \frac{17}{12}$. For the final answer, $\int_0^{2\pi} \frac{\cos 2x dx}{5+4\cos x} = \frac{1}{2i}\, 2\pi i\, (-\frac{5}{4}+\frac{17}{12}) = \pi\frac{2}{12} = \frac{\pi}{6}$.

The Fundamental Theorem of Algebra

Theorem 6.7. If $f(z)$ is analytic at $z_0 \in \mathbb{C}$, then so is $f^{(n)}(z)$ for all $n \in \mathbb{N}$. Writing $f(z) = u(x,y) + iv(x,y)$, this will also imply that both u and v have continuous partial derivatives of all orders at $z_0 = (x_0, y_0)$.

Proof. We can find a sufficiently small circle C with center at z_0 such that $f(z)$ is analytic on and inside C. Then the existence of $f''(z)$ there is given by Theorem 6.4, implying that $f'(z)$ is analytic at z_0, and the proof follows by induction. ▽

Theorem 6.8 (Morera's Theorem)**.** Let $f(z)$ be continuous on a domain D such that $\int_C f(z)dz = 0$ for every closed contour $C \subseteq D$. Then $f(z)$ is analytic in D.

Proof. Theorem 5.5 implies that there exists anti-derivative $F(z)$ such that $F'(z) = f(z)$, i.e., that $F(z)$ is analytic in D, hence so is $f(z)$ according to Theorem 6.7. ▽

Theorem 6.9 (Liouville's Theorem)**.** If a complex function is entire and bounded, then it must be a constant function.

Proof. By Theorem 6.4 we have $f'(z) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}dz$, where we can choose C to be the circle $|z - z_0| = R$, arbitrarily large. Assuming $|f(z)| \leq M$ for all $z \in \mathbb{C}$, we apply Theorem 5.4 to get $|f'(z_0)| \leq \left|\frac{1}{2\pi i}\right| \frac{M}{R^2} 2\pi R = \frac{M}{R} \to 0$ as $R \to \infty$. We conclude that $f'(z_0) = 0$ and so, since z_0 is arbitrary, $f(z)$ is constant by Theorem 4.7. ▽

Theorem 6.10 (The Fundamental Theorem of Algebra)**.** Every non-constant complex polynomial has a zero.

Proof. Let $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ be a complex polynomial with no zeros. We will show that $P(z)$ is constant. Since $f(z) = \frac{1}{P(z)}$ is entire, by Liouville's theorem it suffices to show that $f(z)$ is bounded.

Since $P(z) = z^n(a_0 z^{-n} + a_1 z^{1-n} + \cdots + a_{n-1}z^{-1} + a_n) := z^n(g(z) + a_n)$, where $\lim_{z\to\infty} g(z) = 0$. Hence there exists $\delta > 0$ such that $|P(z)| \geq \delta|z|^n$ whenever $|z|$ is sufficiently large. In particular if $M > 0$ is fixed, there exists $R > 0$ such that $|P(z)| \geq M$ for all z with $|z| > R$. This means that $f(z)$ is bounded on the region outside the circle $|z| = R$. But on the disc $|z| \leq R$, Theorem 3.6 says that $f(z)$ is also bounded, hence bounded on all $\mathbb{C}$. ▽

Exercises

1. Let $z = 5 + 7i$ and $w = 2 + i$. Evaluate and simplify in the form $X + Yi$.
   (a) $z + w$ (b) $z - w$ (c) $z \times w$ (d) $z \div w$
2. Let $z = 2 + 3i$ and $w = -3 + 2i$. Evaluate and simplify in the form $X + Yi$.
   (a) $\overline{z} + w$ (b) $z + \overline{w}$ (c) $\overline{z}w$ (d) $z\overline{w}$
3. Prove that (a) $\overline{z + w} = \overline{z} + \overline{w}$ and (b) $\overline{z \times w} = \overline{z} \times \overline{w}$ for all $z, w \in \mathbb{C}$.
4. Let $z = 5 + 4i$ and $w = i - 2$. Evaluate and simplify in the form $X + Yi$.
   (a) $z^2 - w^2$ (b) $\overline{z}(z - iw)$ (c) zw^{-1} (d) $\overline{w}/iz$
5. Find two complex numbers $z = x + yi$ such that (a) $z^2 = -8 - 6i$ (b) $z^2 = 84i - 13$
6. Solve for $z \in \mathbb{C}$ by factoring, completing the square, or the quadratic formula.
   (a) $z^2 + 2z + 2 = 0$ (b) $iz^2 + 3z + 4i = 0$ (c) $3z^2 - 10iz - 3 = 0$
7. Describe the region in the complex plane with the given condition.
   (a) $|z - 2| \leq 4$ (b) $|2z + 3i| = 5$ (c) $\text{Im}(z) > 1$ (d) $\text{Re}(z^2) \geq 0$
8. Write an equation of the circle centered at z_0 of radius R.
   (a) $z_0 = 3i;\ R = 2$ (b) $z_0 = -5 + 3i;\ R = 7$ (c) $z_0 = \frac{3}{2} - i;\ R = \frac{1}{3}$
9. Convert from rectangular to polar form, where $-\pi < \theta \leq \pi$.
   (a) $3i$ (b) $-2 + 2i\sqrt{3}$ (c) $-5 - 5i$ (d) $\sqrt{3} - i$
10. Convert from polar to rectangular form.
    (a) $(1, 0)$ (b) $(2, -\pi/2)$ (c) $(3, \pi/6)$ (d) $(1, 2\pi/3)$
11. Let $r = (2, \frac{\pi}{2})$ and $s = (3, \frac{\pi}{4})$ in polar form. Evaluate in polar form with $\theta \in (-\pi, \pi]$.
    (a) $r \times s$ (b) $r \times \overline{s}$ (c) r^3s^2 (d) $r^{-2}s^{-1}$
12. Evaluate by converting first to polar form then back to rectangular form.
    (a) $(1 - i)^6$ (b) $(1 + i\sqrt{3})^{10}$ (c) $(-1 + i\sqrt{3})^{-2}$ (d) $(-2 - 2i)^7$
13. Use polar form to find all complex solutions.
    (a) $z^2 = i$ (b) $z^8 = 1$ (c) $z^3 = -8$
14. Evaluate and simplify $(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8})^{14}$ using De Moivre's formula.
15. Evaluate e^z.
    (a) $z = 2 - 3\pi i$ (b) $z = 2 + 3\pi i$ (c) $z = (2 + \pi i)/4$ (d) $z = \pi i$
16. Evaluate $\log z$ and $\text{Log}\, z$.
    (a) $z = e$ (b) $z = -2$ (c) $z = i$ (d) $z = 1 + i\sqrt{3}$ (e) $z = -1 + i\sqrt{3}$
17. Prove that $\text{Log}(1 + i)^2 = 2\,\text{Log}(1 + i)$ but $\text{Log}(-1 + i)^2 \neq 2\,\text{Log}(-1 + i)$.

18. Evaluate using the principal Log.
    (a) i^i (b) $(-1)^{1/\pi}$ (c) $(1-i)^{4i}$ (d) $(-\frac{e}{2} - i\frac{e\sqrt{3}}{2})^{3\pi i}$

19. Evaluate $\cos z$ and $\sin z$.
    (a) $z = \pi - i$ (b) $z = \frac{\pi}{2} + 2i$ (c) $z = \pi i$ (d) $-i$

20. Prove the trigonometric identities.
    (a) $\cos 2z = \cos^2 z - \sin^2 z$ (b) $\sin 2z = 2\sin z \cos z$ (c) $\sin^2 z + \cos^2 z = 1$

21. Prove the trigonometric identities.
    (a) $\cosh(-z) = \cosh z$ (b) $\sinh(-z) = -\sinh z$ (c) $\cosh^2 z - \sinh^2 z = 1$

22. Find a formula for $\cosh(z_1 + z_2)$ and $\sinh(z_1 + z_2)$ with proofs.

23. Find all numbers $z \in \mathbb{C}$ such that (a) $\cosh z = 0$ (b) $\sinh z = 0$.

24. Prove using the definition of limit.
    (a) $\lim_{z\to 1-i}(2z + iz) = 3 - i$ (b) $\lim_{z\to 2+i}\ 3z - 2iz = 8 - i$

25. Prove that $\lim_{z\to 0}(z/\overline{z})^2$ does not exist by showing that the limits along the lines $y = 0$ and $y = x$ are not equal.

26. Prove the limit involving the neigborhood of infinity.
    (a) $\lim_{z\to\infty} \frac{4z^2}{(z-1)^2} = 4$ (b) $\lim_{z\to 1} \frac{1}{(z-1)^3} = \infty$ (c) $\lim_{z\to\infty} \frac{z^2+1}{z-1} = \infty$

27. Find the domain where $f(z)$ is continuous.
    (a) $f(z) = e^{1/z}$ (b) $f = \frac{\cos 2z}{\sin z}$ (c) $f = \text{Log}(z+2)$ (d) $f = \frac{\sin^2 \overline{z}}{z+\overline{z}}$ (e) $f = \frac{1}{z^3+4z}$

28. Evaluate the derivative.
    (a) $f(z) = 3z^2 - 2z + 4$ (b) $f = (1 - 4z^2)^3$ (c) $f = \frac{z-1}{2z+1}$ (d) $f = \frac{(1+z^2)^4}{z^2}$

29. Find the real functions $u(x, y)$ and $v(x, y)$ such that $f(z) = u + iv$.
    (a) $f = \overline{z}^2 + 2iz$ (b) $f = z^3 + z + 1$ (c) $f = \text{Log}\,\overline{z}$ (d) $f = e^{z^2}$ (e) $f = \cos z^2$

30. Use Cauchy-Riemann equations to determine (a) the domain where $f'(z)$ exists and (b) the domain where $f(z)$ is analytic, and (c) find $f'(z)$.

1. $z - \overline{z}$	5. $z\,\text{Im}\,z$	9. $e^{-y}\sin x - ie^{-y}\cos x$
2. $e^x e^{-iy}$	6. $x^3 + i(1-y)^3$	10. $e^y e^{ix}$
3. $e^{-x}e^{-iy}$	7. $2x + ixy^2$	11. $e^x(y^2 + iy - 3i)$
4. $x^2 + iy^2$	8. $3x + y - ix + 3iy$	12. $2x^2 - y^3 + i(x + 2xy - y^2)$

31. Prove the derivative formula for the trigonometric functions.
    (a) $f(z) = \cosh z$ (b) $f(z) = \sinh z$ (c) $f(z) = \tan z$ (d) $f(z) = \cot z$

32. Find the real functions $u(r, \theta)$ and $v(r, \theta)$ such that $f(z) = u + iv$.
    (a) $f(z) = z^3$ (b) $f(z) = z^{1/4}$ (c) $f(z) = z^{-2} + \overline{z}$ (d) $f(z) = \text{Log}\,\overline{z}$

33. Use polar form to prove that $f'(z)$ exists in the given domain and find it.

(a) $f(z) = 1/z^4 \quad (z \neq 0)$
(b) $f(z) = \sqrt{r}e^{i\theta/2} \quad (r > 0, -\pi < \theta < \pi)$
(c) $f(z) = e^{-\theta}\cos(\ln r) + ie^{-\theta}\sin(\ln r) \quad (r > 0, 0 < \theta < 2\pi)$

34. Prove that $u(x, y)$ is harmonic in some domain and find a harmonic conjugate $v(x, y)$.
(a) $u = 2x(1 - y)$ (b) $u = 2x - x^3 + 3xy^2$ (c) $u = \sinh x \sin y$ (d) $u = \frac{y}{x^2+y^2}$

35. Prove for a given domain D.
   (a) If $f(z)$ and $\overline{f(z)}$ are both analytic in D, then f is constant in D.
   (b) If $f(z)$ is analytic in D and $|f(z)|$ is constant in D, then f is constant in D.
   (c) If $v(x, y)$ is a harmonic conjugate of $u(x, y)$ in D and conversely u is a harmonic conjugate of v in D, then both u and v are constant in D.
   (d) In the same domain, v is a harmonic conjugate of u if and only if $-u$ is a harmonic conjugate of v.

36. Evaluate the line integrals.
(a) $\int_0^1 (1 + it)^2\, dt$ (b) $\int_1^2 (\frac{1}{t} - i)^2\, dt$ (c) $\int_0^{\pi/6} e^{i2t}\, dt$

37. Determine $z = z(t)$ and the interval $t \in [a, b]$ for each contour C.
   (a) circle radius 2 centered at $2 - 3i$
   (b) straight line from $-1 + 2i$ to $1 - 2i$
   (c) curve from 0 to $2 + 4i$ along the parabola $y = x^2$
   (d) upper semi-circle radius 1 centered at 1
   (e) straight line from -1 to $2 + 6i$

38. Evaluate the integral $\int_C f(z)\, dz$, where C is the semi-circle $z(t) = e^{it}$ $(0 \leq t \leq \pi)$.
(a) $f(z) = \frac{z+2}{z}$ (b) $f(z) = \overline{z} - 1$ (c) $f(z) = z^2 + z$ (d) $f = z^i$ (principal Log)

39. Evaluate $\int_C \pi e^{\pi\overline{z}}\, dz$ where C is the square in the complex plane with vertices $\{0, 1, 1 + i, i\}$ and positive orientation.

40. Evaluate the contour integral $\int_C f(z)\, dz$ where

$$f(z) = f(x + yi) = \begin{cases} 1 & \text{if } y < 0 \\ 4y & \text{if } y > 0 \end{cases}$$

and C is the arc from $-1 - i$ to $1 + i$ along the curve $y = x^3$.

41. Evaluate using anti-derivative when possible, or using definition otherwise.
   (a) $\int_C e^{\pi z}\, dz$ where C is the straight line from i to $i/2$.
   (b) $\int_C \overline{z}\, dz$ where C is the circle $z(t) = i + e^{it}$ $(0 \leq t \leq 2\pi)$.
   (c) $\int_C \cos\frac{z}{2}\, dz$ where C is the straight line from 0 to $1 + i$ followed by another straight line from $1 + i$ to $\pi + 2i$.
   (d) $\int_C z^{-2}\, dz$ where C is the circle $z(t) = 3e^{it}$ $(-\pi \leq t \leq \pi)$.

(e) $\int_C (z-1)^3\,dz$ where C is the semi-circle $z(t) = i + e^{it}$ $(0 \le t \le \pi)$.

(f) $\int_C \frac{1}{z}\,dz$ where C is the semi-circle $z(t) = 2e^{it}$ for $\frac{\pi}{2} \le t \le \frac{3\pi}{2}$.

(g) $\int_C \frac{1}{z}\,dz$ where C is the semi-circle $z(t) = 2e^{it}$ for $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$.

42. Evaluate using the Cauchy-Goursat Theorem or the Cauchy Integral Formula, where the contour C is the square bounded by $x = \pm 2$ and $y = \pm 2$.

(a) $\displaystyle\int_C \frac{e^{-z}}{z - \frac{\pi i}{2}}\,dz$ (b) $\displaystyle\int_C \frac{\cos z}{z(z^2+8)}\,dz$ (c) $\displaystyle\int_C \frac{z}{2z+1}\,dz$ (d) $\displaystyle\int_C \frac{\cosh z}{z^4}\,dz$

43. Evaluate using the Cauchy-Goursat Theorem or the Cauchy Integral Formula.

1. $\displaystyle\int_{|z|=1} \frac{z^2}{z-3}\,dz$
2. $\displaystyle\int_{|z|=1} \frac{1}{z^2+2z+2}\,dz$
3. $\displaystyle\int_{|z|=3} \frac{2z^2-z-2}{z+4i}\,dz$
4. $\displaystyle\int_{|z-i|=2} \frac{1}{z^2+4}\,dz$
5. $\displaystyle\int_{|z|=1} ze^{-z}\,dz$
6. $\displaystyle\int_{|z|=1} \operatorname{Log}(z+2)\,dz$
7. $\displaystyle\int_{|z|=2} \frac{1}{(z^2+1)^2}\,dz$
8. $\displaystyle\int_{|z|=5} \frac{z^3+2z}{(z+3)^3}\,dz$
9. $\displaystyle\int_{|z|=2} \frac{1}{z^2+2z+2}\,dz$
10. $\displaystyle\int_{|z-2i|=3} \frac{1}{z^3+4z}\,dz$
11. $\displaystyle\int_{|z-i|=2} \frac{1}{(z^2+4)^2}\,dz$

44. Evaluate the improper real integrals using Cauchy Integral Formula.

1. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^3}$
2. $\displaystyle\int_{0}^{\infty} \frac{dx}{(1+x^2)^2}$
3. $\displaystyle\int_{0}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)}$
4. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2-2x+5}$
5. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$
6. $\displaystyle\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+1)(x^2+2x+2)}$

45. Evaluate the real definite integrals using Cauchy Integral Formula.

1. $\displaystyle\int_{-\pi}^{\pi} \frac{dx}{5-3\sin x}$
2. $\displaystyle\int_{-\pi}^{\pi} \frac{dx}{5+4\sin x}$
3. $\displaystyle\int_{0}^{\pi} \frac{\cos 2x\,dx}{5+4\cos x}$
4. $\displaystyle\int_{0}^{\pi} \frac{\cos 3x\,dx}{5-4\cos x}$
5. $\displaystyle\int_{0}^{2\pi} \frac{\cos 2x\,dx}{5-3\cos x}$
6. $\displaystyle\int_{0}^{2\pi} \frac{dx}{(5-3\sin x)^2}$
7. $\displaystyle\int_{-\pi}^{\pi} \frac{dx}{(3+\cos x)^2}$
8. $\displaystyle\int_{0}^{\pi} \frac{dx}{1+\sin^2 x}$
9. $\displaystyle\int_{0}^{2\pi} \frac{dx}{3-2\cos x+\sin x}$
10. $\displaystyle\int_{0}^{2\pi} \frac{\cos^2 3x\,dx}{5-4\cos 2x}$

Answers

1. (a) $7+8i$ (b) $3+6i$ (c) $3+19i$ (d) $\frac{17}{\sqrt{5}}+\frac{9}{\sqrt{5}}i$
2. (a) $-1+i$ (b) $-1-i$ (c) $13i$ (d) $-13i$
3. Let $z=a+bi$ and $w=c+di$.
4. (a) $6+44i$ (b) $54+6i$ (c) $-\frac{1}{5}(6+13i)$ (d) $\frac{1}{41}(3+14i)$
5. (a) $\pm(1-3i)$ (b) $\pm(6+7i)$
6. (a) $-1\pm i$ (b) $-i, 4i$ (c) $\frac{i}{3}, 3i$
7. (a) closed disc at $(2,0)$ radius 4 (b) circle at $(0,-\frac{3}{2})$ radius 2.5 (c) half-plane above $y=1$ (d) region bounded by the lines $y=\pm x$
8. (a) $|z-3i|=2$ (b) $|z+5-3i|=7$ (c) $|6z-9+6i|=2$
9. (a) $(3,\frac{\pi}{2})$ (b) $(4,\frac{2\pi}{3})$ (c) $(5\sqrt{2},-\frac{3\pi}{4})$ (d) $(2,-\frac{\pi}{6})$
10. (a) 1 (b) $-2i$ (c) $\frac{3}{2}(\sqrt{3}+i)$ (d) $-\frac{1}{2}+\frac{i}{2}\sqrt{3}$
11. (a) $(6,\frac{3\pi}{4})$ (b) $(6,\frac{\pi}{4})$ (c) $(72,0)$ (d) $(\frac{1}{12},\frac{3\pi}{4})$
12. (a) $8i$ (b) $-512-512i\sqrt{3}$ (c) $-\frac{1}{8}+\frac{i}{8}\sqrt{3}$ (d) $-1024+1024i$
13. (a) $(1,\frac{\pi}{4}),(1,-\frac{3\pi}{4})$ (b) $(1,\frac{k\pi}{4}); k=0,\ldots,7$ (c) $(2,\frac{\pi}{3}),(2,\pi),(2,\frac{5\pi}{3})$
14. $\frac{1-i}{\sqrt{2}}$
15. (a) $-e^2$ (b) $-e^2$ (c) $\sqrt{\frac{e}{2}}+i\sqrt{\frac{e}{2}}$ (d) -1
16. (a) $1+2n\pi i$ (b) $\ln 2+i(\pi+2n\pi)$ (c) $i(\frac{\pi}{2}+2n\pi)$ (d) $\ln 2+i(\frac{\pi}{3}+2n\pi)$ (e) $\ln 2+i(\frac{2\pi}{3}+2n\pi)$
17. (a) $\ln 2+i\frac{\pi}{2}$ (b) $\ln 2+i\frac{3\pi}{2}\neq\ln 2-i\frac{\pi}{2}$
18. (a) $e^{-\pi/2}$ (b) $\cos 1+i\sin 1$ (c) $e^{\pi}\cos\ln 4+ie^{\pi}\sin\ln 4$ (d) $-e^{2\pi^2}$
19. (a) $\frac{e+e^{-1}}{-2}$; $(\frac{e^{-1}-e}{2})i$ (b) $(\frac{e^{-2}-e^{2}}{2})i$; $\frac{e^2+e^{-2}}{2}$ (c) $\frac{e^{\pi}+e^{-\pi}}{2}$; $(\frac{e^{\pi}-e^{-\pi}}{2})i$ (d) $\frac{e+e^{-1}}{2}$; $(\frac{e^{-1}-e}{2})i$
20. Use Theorem 2.4
21. Use Theorem 2.7
22. (a) $\cosh z_1\cosh z_2+\sinh z_1\sinh z_2$
    (b) $\sinh z_1\cosh z_2+\cosh z_1\sinh z_2$
23. (a) $(\frac{\pi}{2}+n\pi)i$ (b) $n\pi i$
24. (a) Let $\delta=\frac{\varepsilon}{3}$ (b) Let $\delta=\frac{\varepsilon}{5}$
25. $1\neq -1$
26. Use Theorem 3.3

27. (a) $\{z \neq 0\}$ (b) $\mathbb{C} \setminus \{\pi n \mid n \in \mathbb{Z}\}$ (c) $\mathbb{C} \setminus (-\infty, -2]$ (d) $\{\operatorname{Re} z \neq 0\}$ (5e) $\{z \neq 0, \pm 2i\}$

28. (a) $6z - 2$ (b) $-24z(1 - 4z^2)^2$ (c) $3(2z + 1)^{-2}$ (d) $2(3z^2 - 1)(1 + z^2)^3 z^{-3}$

29. (a) $u = x^2 - y^2 - 2y;\ v = 2x - 2xy$ (b) $u = x^3 - 3xy^2 + x + 1;\ v = -y^3 + 3x^2y + y$ (c) $u = \frac{1}{2}\ln(x^2 + y^2);\ v = -\arctan(y/x)$ (d) $u = e^{x^2-y^2}\cos 2xy; v = e^{x^2-y^2}\sin 2xy$ (e) $u = \cos(x^2 - y^2)\cosh(2xy);\ v = \sin(y^2 - x^2)\sinh(2xy)$

30. (1) $\emptyset$ (2) $\emptyset$ (3) $f'(z) = -e^{-z}\ \forall z$ (4) $f'(x + xi) = 2x$ (5) $f'(0) = 0$ (6) $f'(i) = 0$ (7) $\emptyset$ (8) $f'(z) = 3 - i\ \forall z$ (9) $f'(z) = e^{iz}\ \forall z$ (10) $\emptyset$ (11) $f'(x + i) = e^x(1 - 2i)$ (12) $f'(\frac{1}{3}, -\frac{1}{3}) = \frac{4}{3} + \frac{i}{3}; f'(-1, 1) = 3i - 4$

31. (a) $\sinh z$ (b) $\cosh z$ (c) $\sec^2 z$ (d) $-\csc^2 z$

32. (a) $u = r^3\cos 3\theta;\ v = r^3 \sin 3\theta$ (b) $u = \sqrt[4]{r}\cos\frac{\theta}{4};\ v = \sqrt[4]{r}\sin\frac{\theta}{4}$ (c) $u = \frac{1}{r^2}\cos 2\theta + r\cos\theta;\ v = -\frac{1}{r^2}\sin 2\theta - r\sin\theta$ (d) $u = \ln r;\ v = -\theta$

33. (a) $f' = -\frac{4}{z^5}$ (b) $f' = \frac{1}{2f}$ (c) $f' = \frac{if}{z}$

34. (a) $x^2 - y^2 + 2y$ (b) $2y - 3x^2y + y^3$ (c) $-\cosh x \cos y$ (d) $\frac{x}{x^2+y^2}$

35. Use Theorem 4.4

36. (a) $\frac{2}{3} + i$ (b) $-\frac{1}{2} - i\ln 4$ (c) $\frac{1}{4}(\sqrt{3} + i)$

37. (a) $2 - 3i + 2e^{it}$ $(0 \le t \le 2\pi)$ (b) $t - 2it$ $(-1 \le t \le 1)$ (c) $t + it^2$ $(0 \le t \le 2)$ (d) $1 + e^{it}$ $(0 \le t \le \pi)$ (e) $t + i(2t + 2)$ $(-1 \le t \le 2)$

38. (a) $2\pi i - 2$ (b) $2 + \pi i$ (c) $-\frac{2}{3}$ (d) $-(\frac{1+e^{-\pi}}{2}) + i(\frac{1+e^{-\pi}}{2})$

39. $4e^{\pi} - 4$

40. $2 + 3i$

41. (1) $\frac{1}{\pi} + \frac{i}{\pi}$ (2) $2\pi i$ (3) $\frac{e^2+1}{e}$ (4) 0 (5) $-2 - 6i$ (6) πi (7) πi

42. (a) 2π (b) $\frac{\pi}{4}i$ (c) $-\frac{\pi}{2}i$ (d) 0

43. (1) 0 (2) 0 (3) 0 (4) $\frac{\pi}{2}$ (5) 0 (6) 0 (7) 0 (8) $-18\pi i$ (9) 0 (10) $\frac{\pi}{4}i$ (11) $\frac{\pi}{16}$

44. (1) $\frac{\pi}{2}$ (2) $\frac{2\pi}{3}$ (3) $\frac{\pi}{6}$ (4) π (5) $\frac{\pi}{12}$ (6) $\frac{5\pi}{32}$

45. (1) $\frac{\pi}{\sqrt{8}}$ (2) $\frac{\pi}{4}$ (3) $\frac{\pi}{3}$ (4) $\frac{3\pi}{256}$ (5) π (6) $-\frac{\pi}{5}$