Discrete Structures†

Revision Notes and Problems

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Preface

These notes were prepared for students as a revision workbook and are not meant to substitute the in-class notes. No student is expected to really benefit from these notes unless they have regularly attended the lectures.

Affordable Texts

1. E. A. Bender and S. G. Williamson, A Short Course in Discrete Mathematics, Dover 2005.
2. S. G. Krantz, Discrete Mathematics Demystified, McGraw-Hill 2008.
3. S. Lipschutz and M. Lipson, Discrete Mathematics: Schaum's Outlines, McGraw-Hill 2009.
4. A. Witno, Discrete Structures in Five Chapters, CreateSpace 2010.

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† Last updated 25–05–2016

Chapter 1
Propositional Logic

A **proposition** is a statement which has a truth value: either true or false.

Examples: 1) Amman is in Jordan
2) $2 + 2 = 4$
3) $2 + 2 = 5$

Some statements are not a proposition because they have no truth values.

Examples: 1) Philadelphia University
2) $n + n = 2n$
3) $x + y = 0$

The **negation** of a proposition p (not p) is denoted by $\neg p$.

Examples: 1) p: Amman is in Jordan
$\neg p$: Amman is not in Jordan
2) p: $2 + 2 = 5$
$\neg p$: $2 + 2 \neq 5$

The **conjunction** of two propositions: $p \wedge q$ (p and q) is one whose value is true only when both are true. The **disjunction** $p \vee q$ (p or q) is false only when both are false.

1.1 Let p: Amman is in Jordan and q: $2 + 2 = 5$.
a) What is the proposition $p \wedge \neg q$?
b) What is the value of $p \wedge \neg q$?
c) What is the proposition $\neg p \vee \neg q$?
d) What is the value of $\neg p \vee \neg q$?

The **implication** of two propositions: $p \rightarrow q$ (if p then q) is one whose value is false only when p is true and q is false. The **biconditional** $p \leftrightarrow q$ (p if and only if q) is true only when the values of p and q are the same, whereas the **exclusive or** $p \oplus q$ (either p or q but not both) is true only when the values of p and q are not the same.

1.2 Let p: Today is cold, q: Today is hot, and r: Today is windy. Write the following propositions using p, q, and r.
a) Today is hot if and only if not windy.
b) Either today is cold or not cold, but not both.
c) If today is not windy then it is not hot.
d) Today is neither cold nor windy.
e) If today is windy then either it is hot or cold.

Logic operators can be presented in their **truth tables**:

p	q	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$	$p \oplus q$
T	T	T	T	T	T	F
T	F	F	T	F	F	T
F	T	F	T	T	F	T
F	F	F	F	T	T	F

1.3 Draw the truth table for each of the following propositions.
a) $\neg p \vee \neg q$
b) $\neg(p \wedge q) \rightarrow p$
c) $(p \oplus \neg q) \leftrightarrow (\neg p \vee q)$
d) $(p \rightarrow q) \rightarrow r$
e) $[(p \wedge q) \rightarrow r] \oplus [\neg p \vee (q \leftrightarrow r)]$

Two propositions are **equivalent** if their truth tables are identical, for example exclusive or is equivalent to the negation of biconditional: $p \oplus q \equiv \neg(p \leftrightarrow q)$

1.4 Prove the following equivalences by drawing the truth tables.

a) $\neg p \vee \neg q \equiv \neg(p \wedge q)$
b) $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
c) $p \rightarrow (q \rightarrow r) \equiv q \rightarrow (p \rightarrow r)$

1.5 The **difference** of two propositions is defined by $p - q \equiv p \wedge \neg q$. Prove that

$p \rightarrow q \equiv \neg(p - q)$.

The **contrapositive** of $p \rightarrow q$ is the proposition $\neg q \rightarrow \neg p$. It is not difficult to show that an implication is equivalent to its contrapositive: $p \rightarrow q \equiv \neg q \rightarrow \neg p$.

1.6 For each proposition below write an equivalent statement using contrapositive.

a) If I study hard then I get good mark.
b) If it rains then it is not hot.
c) If today is not Sunday then tomorrow is not Monday.
d) If I am not lazy then I come to the lecture.

A **tautology** is a compound proposition whose truth table is all true, whereas a **contradiction** is all false. A **contingency**.is a mix of true and false.

1.7 Identify each proposition as a tautology, contradiction, or contingency.

a) $(p \wedge q) \rightarrow p$
b) $p \rightarrow (p \vee q)$
c) $p \rightarrow (p \rightarrow q)$
d) $p \rightarrow (q \rightarrow p)$
e) $\neg p \wedge \neg(p \rightarrow q)$

1.8 An **argument** consists of two components: a set of **premises** $p_1, p_2, \ldots, p_n$ and a **conclusion** Q. The argument (or its conclusion) is **valid** if $p_1 \wedge p_2 \wedge \ldots \wedge p_n \rightarrow Q$ is a tautology. Which of the following arguments are valid?

a) p_1: I failed my exam today
   p_2: If I studied last night then I did not fail my exam today
   Q: I did not study last night
b) p_1: If it snows then the school is closed
   p_2: It is not snowing
   Q: The school is not closed

The following is a list of some common logical equivalence rules:

1) $p \wedge q \equiv q \wedge p$
   $p \vee q \equiv q \vee p$
2) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
   $p \vee (q \vee r) \equiv (p \vee q) \vee r$
3) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
   $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4) $\neg(\neg p) \equiv p$
   $\neg(p \wedge q) \equiv \neg p \vee \neg q$
   $\neg(p \vee q) \equiv \neg p \wedge \neg q$
5) $p \rightarrow q \equiv \neg p \vee q$
   $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
   $p \oplus q \equiv \neg(p \leftrightarrow q)$

1.9 Prove by applying the above rules.

a) $\neg(p \rightarrow q) \equiv p \wedge \neg q$
b) $p \rightarrow q \equiv \neg q \rightarrow \neg p$
c) $p \rightarrow (q \rightarrow r) \equiv q \rightarrow (p \rightarrow r)$
d) $p \rightarrow (q \wedge r) \equiv (p \rightarrow q) \wedge (p \rightarrow r)$
e) $p \oplus q \equiv (p \wedge \neg q) \vee (q \wedge \neg p)$

1.10 True or False. Prove by any method you like.

a) $p \to (q \to r) \equiv (p \to q) \to r$
b) $p \to (q \vee r) \equiv (p \to q) \vee (p \to r)$
c) $p \vee (q \oplus r) \equiv (p \vee q) \oplus (p \vee r)$
d) $\neg(p \oplus q) \equiv \neg p \leftrightarrow \neg q$

A **CNF** (Conjunctive Normal Form) is a compound proposition in the form conjunctions of disjunctions of propositional variables or their negations, for example $(p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee \neg q)$. Similarly a **DNF** (Disjunctive Normal Form) is disjunctions of conjunctions, such as $(p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge \neg r)$. We say that the normal form is **full** when no variable is missing in each bracket.

Theorem: Every compound proposition is equivalent to a CNF and to a DNF.

Example: Convert $[(p \leftrightarrow q) \oplus \neg p] \to \neg q$ to a CNF and to a DNF.
Solution: First draw the truth table. The result is

p	q	…	$[(p \leftrightarrow q) \oplus \neg p] \to \neg q$
T	T	…	F
T	F	…	T
F	T	…	F
F	F	…	T

A full CNF can be obtained by selecting the variables with false values from each row of the table whose result is false: $(\neg p \vee \neg q) \wedge (p \vee \neg q)$ and similarly a full DNF from the true: $(p \wedge \neg q) \vee (\neg p \wedge \neg q)$. Both forms are equivalent to the given proposition: $[(p \leftrightarrow q) \oplus \neg p] \to \neg q \equiv (\neg p \vee \neg q) \wedge (p \vee \neg q) \equiv (p \wedge \neg q) \vee (\neg p \wedge \neg q)$.

1.11 Convert each proposition to a CNF and to a DNF.

a) $\neg(p \wedge q) \to p$
b) $(p \oplus \neg q) \leftrightarrow (\neg p \vee q)$
c) $(p \to q) \to r$
d) $[(p \wedge q) \to r] \oplus [\neg p \vee (q \leftrightarrow r)]$

1.12 Convert each CNF to DNF and vice versa.

a) $(p \wedge q) \vee (\neg p \wedge q)$
b) $(p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee \neg q)$
c) $(p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge \neg r)$
d) $(p \vee \neg q \vee r) \wedge (\neg p \vee q \vee r)$

Chapter 2
Methods of Proof

Direct Proof:

To prove a proposition in the form $p \to q$, we begin by assuming that p is true and then show that q must be true.

Example: An **even** number is of the form 2n where n is an integer, whereas an **odd** number is $2n + 1$. Prove that if x is an odd integer then x^2 is also odd.

Solution: Let p: x is odd, and q: x^2 is odd. We want to prove $p \to q$.
Start: p: x is odd
$\to x = 2n + 1$ for some integer n
$\to x^2 = (2n + 1)^2$
$\to x^2 = 4n^2 + 4n + 1$
$\to x^2 = 2(2n^2 + 2n) + 1$
$\to x^2 = 2m + 1$, where $m = (2n^2 + 2n)$ is an integer
$\to x^2$ is odd
$\to q$

2.1 Prove the following propositions.
a) If x is an even integer then x^3 is also even.
b) If x is an odd integer then x^3 is also odd.
c) If x and y are odd integers then $x + y$ is even.
d) If x and y are odd integers then xy is also odd.
e) If x is an odd integer then $x^2 - 3x$ is even.

Proof by Contrapositive:

To prove a proposition in the form $p \to q$, we may instead prove its contra-positive: $\neg q \to \neg p$. This works because $p \to q \equiv \neg q \to \neg p$.

Example: Prove that if x^2 is odd then x must be odd.

Solution: Let p: x^2 is odd, and q: x is odd. We will prove $p \to q$ by proving $\neg q \to \neg p$.
Start: $\neg q$: x is even
$\to x = 2n$ for some integer n
$\to x^2 = (2n)^2$
$\to x^2 = 4n^2$
$\to x^2 = 2(2n^2)$
$\to x^2 = 2m$, where $m = 2n^2$ is an integer
$\to x^2$ is even
$\to \neg p$

2.2 Prove the following propositions.
a) If x^2 is even then x must be even.
b) If x^3 is even then x must be even.
c) If $x^2 - 2x$ is even then x must be even.
d) If $x^3 - 4x + 2$ is odd then x must be odd.

Proving Equivalence:

To prove a proposition in the form $p \leftrightarrow q$, we prove its equivalence:
$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$.

Example: Prove that x^2 is odd if and only if x is odd.

Solution: Let p: x^2 is odd, and q: x is odd. We will prove $p \leftrightarrow q$ by proving $p \to q$ and $q \to p$
Step 1) Prove $p \to q$ ... (Like Example 2)
Step 2) Prove $q \to p$ ... (Like Example 1)
Proof is complete.

2.3 Prove the following propositions.

a) x^2 is even if and only if x is even.
b) x^3 is even if and only if x is even.
c) $x^2 - 2x$ is even if and only if x is even.
d) $x^3 - 4x + 2$ is odd if and only if x is odd.

A **predicate** is a propositional function such as P(x): $x + 2 = 5$. For each value of x, P(x) becomes a proposition, for instance, P(3): $3 + 2 = 5$ is true and P(2): $2 + 2 = 5$ is false.

2.4 Let P(x): $x^2 < x$.

a) What is the value of P(1)?
b) What is the value of P(2)?
c) For which x is the value of P(x) true?

2.5 Let P(x,y): $x^2 + y^2 = (x + y)^2$. Find the values of the following propositions.

a) P(0,1)
b) P(0,0)
c) P(1,1)
d) For which (x,y) is the value of P(x,y) true?

A predicate can also be made a proposition by adding a **quantifier** such as $\exists$ (there is / there exists / there is at least one) and $\forall$ (for all / for any /for each).

Example: Let P(x): $x + 2 = 5$.

1) $\exists x\, P(x)$ means "there is at least one x such that $x + 2 = 5$" which is true.
2) $\forall x\, P(x)$ means "for all x, $x + 2 = 5$" which is false.

2.6 Let P(x): $x < 2x$.

a) What is the value of $\exists x\, P(x)$?
b) What is the value of $\forall x\, P(x)$?

2.7 Let P(x,y): $x^2 + y^2 = (x + y)^2$. Find the values of the following propositions.

a) $\exists x\, \exists y\, P(x,y)$
b) $\forall x\, \forall y\, P(x,y)$
c) $\exists x\, \forall y\, P(x,y)$
d) $\forall x\, \exists y\, P(x,y)$
e) $\exists y\, \forall x\, P(x,y)$

2.8 Repeat Problem 1.17 using the following predicates.

a) P(x,y): $x^2 + y^2 > 0$
b) P(x,y): $x^2 + y^2 \geq 0$
c) P(x,y): $x^2 - y^2 \geq 0$

Mathematical Induction:

To prove a proposition in the form $\forall n\, P(n)$ where n is a positive integer, it suffices to prove the following two propositions.

1) P(1)
2) $P(n) \rightarrow P(n+1)$

Example: Prove the following formula for all positive integers n.
$1 + 3 + 5 + 7 + 9 + \ldots + (2n - 1) = n^2$

Solution: Let P(n): $1 + 3 + 5 + 7 + 9 + \ldots + (2n - 1) = n^2$
We shall prove $\forall n\, P(n)$ in two steps:

1) P(1): $1 = 1^2$ so this proposition is true.

2) P(n): $1 + 3 + 5 + 7 + 9 + \dots + (2n - 1) = n^2$
$\rightarrow$ $1 + 3 + 5 + 7 + 9 + \dots + (2n - 1) + (2n + 1) = n^2 + (2n + 1)$
$\rightarrow$ $1 + 3 + 5 + 7 + 9 + \dots + (2n - 1) + (2n + 1) = (n + 1)^2$
$\rightarrow$ P(n+1)

2.9 Prove the following formulas for all positive integers n.

a) $1 + 2 + 3 + 4 + 5 + \dots + n = n(n + 1) \div 2$
b) $2 + 4 + 6 + 8 + 10 + \dots + 2n = n^2 + n$
c) $1 + 2 + 4 + 8 + 16 + \dots + 2^{n-1} = 2^n - 1$
d) $1 + 3 + 9 + 27 + 81 + \dots + 3^{n-1} = (3^n - 1) \div 2$
e) $1 + 4 + 9 + 16 + 25 + \dots + n^2 = n(n + 1)(2n + 1) \div 6$

2.10 Prove the following propositions.

a) $n < 2^n \ \forall\, n \geq 1$
b) $2^n < n! \ \forall\, n \geq 4$
c) $3^n < n! \ \forall\, n \geq 7$
d) $2^n > n^2 \ \forall\, n \geq 5$
e) $n! < n^n \ \forall\, n \geq 2$

Chapter 3
The Integers

In the **binary** number system we use only 0 and 1 to count. The positive integers go like this: 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111, …

Note that the digits of a decimal number represent powers of 10, for example $59012 = 5 \times 10^4 + 9 \times 10^3 + 0 \times 10^2 + 1 \times 10^1 + 2 \times 10^0$. Similarly, the digits of a binary number represent powers of 2.

Examples: 1) $1101_2 = 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 8 + 4 + 0 + 1 = 13_{10}$
2) $101110_2 = 2^1 + 2^2 + 2^3 + 2^5 = 2 + 4 + 8 + 32 = 46_{10}$

3.1 Convert these binary numbers to decimal.
a) 101010
b) 101001000
c) 10110111
d) 1000001

Example: Convert the number 54 to binary.
Solution: To find the appropriate powers of 2, divide this number by 2 repeatedly.
$54 \div 2 = 27$ remain 0
$27 \div 2 = 13$ remain 1
$13 \div 2 = 6$ remain 1
$6 \div 2 = 3$ remain 0
$3 \div 2 = 1$ remain 1
$1 \div 2 = 0$ remain 1
The answer is these remainders from the last one up: 110110

3.2 Convert these decimal numbers to binary.
a) 37
b) 99
c) 500
d) 999

The **hexadecimal** number system uses 16 digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F. Again, the digits represent powers of the base, in this case 16.

Example: $1A5E_{16} = 1 \times 16^3 + 10 \times 16^2 + 5 \times 16^1 + 14 \times 16^0 = 4096+2560+80+14 = 6750_{10}$

3.3 Convert these hexadecimal numbers to decimal.
a) 5FE
b) BCD
c) A0A0
d) 11111

3.4 Convert the decimal numbers in Problem 3.2 to hexadecimal.

Because $16 = 2^4$, every 1 hexadecimal digit corresponds to 4 binary digits.

Examples: 1) $7BF8_{16} = 0111\ 1011\ 1111\ 1000 = 111101111111000_2$
2) $1111011111000101101002_2 = 0011\ 1101\ 1111\ 1000\ 1011\ 0100 = 3DF8B4_{16}$

The following table is useful when converting between binary and hexadecimal.

0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
0000	0001	0010	0011	0100	0101	0110	0111	1000	1001	1010	1011	1100	1101	1110	1111

3.5 Convert the hexadecimal numbers in Problem 3.3 to binary.

3.6 Convert the binary numbers in Problem 3.1 to hexadecimal.

3.7 Convert the decimal numbers in Problem 3.2 to **octal** (base 8).

3.8 A real number between 0 and 1 is represented by negative powers of the base. For example, in decimal $0.125 = 1 \times 10^{-1} + 2 \times 10^{-2} + 5 \times 10^{-3}$. Convert the following numbers to decimal.

a) 0.1101_2
b) 0.000001_2
c) 111.111_2
d) $0.A8_{16}$
e) 111.111_{16}

3.9 Convert these decimal numbers to binary and then to hexadecimal.

a) 0.03125
b) 0.765625
c) 5/8
d) 1/3
e) 25.25

The **floor** function $\lfloor x \rfloor$ of a real number x is the greatest integer $n \le x$ whereas the **ceiling** function $\lceil x \rceil$ is the smallest integer $n \ge x$.

Examples:
1) $\lfloor 1.99 \rfloor = 1$ $\qquad \lceil 1.99 \rceil = 2$
2) $\lfloor 7.01 \rfloor = 7$ $\qquad \lceil 7.01 \rceil = 8$
3) $\lfloor 5 \rfloor = 5$ $\qquad \lceil 5 \rceil = 5$
4) $\lfloor -½ \rfloor = -1$ $\qquad \lceil -½ \rceil = 0$

For two integers m and n > 0 define the modulo operation m **mod** n = $m - \lfloor m/n \rfloor \times n$ which is the same as the remainder upon dividing m by n.

Example: Evaluate 217 mod 5.
Solution: $217 \div 5 = 43.4$ hence 217 mod 5 = $217 - (43 \times 5) = 2$.
Equivalently, $217 = (43) \times 5 + (2)$ hence 217 mod 5 = 2.

3.10 Evaluate the following.

a) 123 mod 3
b) 2000 mod 7
c) 25 mod 11
d) 11 mod 25

In the special case where **m mod n = 0** we say that m is a **multiple** of n and that n is a **divisor** of m.

Examples:
a) 3 is a divisor of 12 and 21 but not of 32
b) even numbers are multiples of 2
c) 0 is a multiple of all integers except itself

3.11 Prove that if a mod n = b mod n then n is a divisor of a – b.

3.12 Prove by induction for all positive integers n.

a) $2^{2n} - 1$ is a multiple of 3
b) 7 is a divisor of $2^{3n} - 1$
c) $n^3 + 2n$ is a multiple of 3
d) $n^5 - n$ mod 5 = 0
e) $2^{n+2} + 3^{2n+1}$ is a multiple of 7

The **GCD** (greatest common divisor) of two integers is the biggest integer that is a divisor of both. Similarly the **LCM** (least common multiple) is the smallest integer a multiple of both.

Examples: GCD (12, 16) = 4 since 4 is a divisor of 12 and 16 and is the biggest of such.
LCM (12, 16) = 48 since 48 is a multiple of 12 and 16 and the smallest of such.

The **Euclidean algorithm** gives an efficient way to compute GCD by iteration:

GCD (m, n) := GCD (n, m mod n)

Example: Find GCD (278, 144) using the algorithm.

Solution: GCD (278, 144)

= GCD (144, 134)	because 278 mod 144 = 134
= GCD (134, 10)	because 144 mod 134 = 10
= GCD (10, 4)	because 134 mod 10 = 4
= GCD (4, 2)	because 10 mod 4 = 2
= GCD (2, 0)	because 4 mod 2 = 0
= 2	

The sequence of remainders consists of
278, 144, 134, 10, 4, 2, 0.

3.13 Find the GCD of each pair using the Euclidean algorithm.
a) 275 and 115
b) 999 and 123
c) 456 and 144
d) 725 and 1000

Theorem: CGD (m, n) $\times$ LCM (m, n) = m $\times$ n

3.14 Find the LCM of each pair in Problem 3.13.

A **sequence** is a function f(n) defined over the (non-negative) integers, hence can be ordered f(0), f(1), f(2), f(3), …

Examples: 1) $f(n) = n^2$ is the sequence 0, 1, 4, 9, 16, 25, 36, …
2) f(n) = 2n + 1 is the sequence 1, 3, 5, 7, 9, 11, 13, …

A sequence is **recursive** if f(n) depends on f(0), f(1), … , f(n–1).

Example: The **Fibonacci** sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, … is recursive with a **recurrence relation** given by f(n) = f(n–1) + f(n–2) for all $n \geq 2$.

3.15 Find a recurrence relation for each given sequence.
a) 1, 3, 5, 9, 17, 31, 57, 105, …
b) 7, 17, 27, 37, 47, 57, 67, …
c) 1, 1, 2, 6, 24, 120, 720, …
d) 2, 4, 5, 7, 9, 12, 16, 22, …

A recurrence relation of the form f(n) = A f(n–1) + B f(n–2) can be expressed explicitly in one of two ways, depending whether the quadratic equation $x^2 - Ax - B = 0$ has one solution or two, respectively:

If there is only one solution (x) then
1) $f(n) = C x^n + D nx^n$

If there are two solutions (x_1 and x_2) then
2) $f(n) = C x_1^n + D x_2^n$

Example: Find an explicit formula for the sequence given by
$f(0) = 4$, $f(1) = 7$, $f(n) = f(n-1) + 6f(n-2)$ for all $n \geq 2$.

Solution: The equation $x^2 - x - 6 = 0$ has two solutions $x_1 = -2$ and $x_2 = 3$ (How?)
Hence the explicit formula is $f(n) = C(-2)^n + D(3)^n$
To find C and D substitute the values of f(0) and f(1):
$f(0) = 4 = C + D$
$f(1) = 7 = -2C + 3D$
The solution is $C = 1$ and $D = 3$ (How?) therefore $f(n) = (-2)^n + 3^{n+1}$.

3.16 Find an explicit formula for each given sequence.

a) $f(0) = 1$, $f(1) = 8$, $f(n) = f(n-1) + 2f(n-2)$
b) $f(0) = 1$, $f(1) = 3$, $f(n) = 4f(n-1) - 4f(n-2)$
c) $a_0 = 1$, $a_1 = 2$, $a_n = 2a_{n-1} + 3a_{n-2}$
d) $a_0 = 1$, $a_1 = 4$, $a_n = 2a_{n-1} - a_{n-2}$

3.17 Find an explicit formula for the Fibonacci sequence.

3.18 Prove that GCD $[f(n), f(n+1)] = 1$ for all $n \geq 0$ in the Fibonacci sequence.

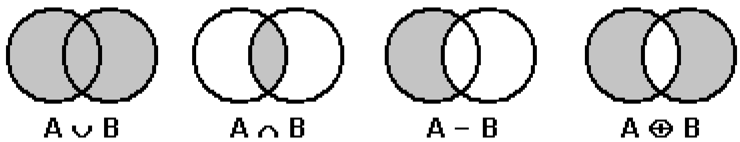
Chapter 4
Sets and Counting

A **set** is a collection of objects called the **elements** of the set. The ordering of the elements is not important and repetition of elements is ignored, for example {1, 3, 1, 2, 2, 1} = {1, 2, 3}. A set may also be empty and it is denoted by ϕ or { }. If x is an element of the set A then we write $x \in A$, otherwise $x \notin A$.

For any two sets A and B, define the following set operations.

1) The **union** $A \cup B = \{x \mid x \in A \vee x \in B\}$
2) The **intersection** $A \cap B = \{x \mid x \in A \wedge x \in B\}$
3) The **difference** $A - B = \{x \mid x \in A \wedge x \notin B\}$
4) The **symmetric difference** $A \oplus B = \{x \mid x \in A \oplus x \in B\}$

These set operations can be illustrated using **Venn diagrams:**



Example: If A = {1, 2, 3, 4, 5} and B = {0, 2, 4, 6} then
$A \cup B = \{0, 1, 2, 3, 4, 5, 6\}$
$A \cap B = \{2, 4\}$
$A - B = \{1, 3, 5\}$
$B - A = \{0, 6\}$
$A \oplus B = \{0, 1, 3, 5, 6\}$

4.1 Let A = {1, 2, 3, 4, 5}, B = {0, 2, 4, 6} and C = {1, 3, 5}. Find the following set.
a) $(A \cup C) \oplus (A \cap C)$
b) $A \oplus (B \cup C)$
c) $(A \oplus B) - (A \oplus C)$
d) $(A - B) \oplus (A - C)$

Define the **complement** of a set A to be $A^C = \{x \mid x \notin A\}$. The following set identities are the analog of logical equivalences.

$(A^C)^C = A$	$A - B = A \cap B^C$
$A \cup B = B \cup A$	$A \cup (B \cup C) = (A \cup B) \cup C$
$A \cap B = B \cap A$	$A \cap (B \cap C) = (A \cap B) \cap C$
$(A \cup B)^C = A^C \cap B^C$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$(A \cap B)^C = A^C \cup B^C$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

4.2 True or False? Use Venn diagrams to verify each one.
a) $(A \cup B) - (A \cap B) = A \oplus B$
b) $(A - B) \cup (B - A) = A \oplus B$
c) $(A \oplus B) - B = A$
d) $(A \oplus B) \oplus B = A$
e) $A \oplus A = A - A$

A set S is a **subset** of a set A, written $S \subseteq A$, if $x \in S \rightarrow x \in A$. For example A = {1, 2} has a total of 4 subsets: {1}, {2}, ϕ, and A itself. The **power set** of a set A, written P(A), is the set whose elements are all the subsets of A. So for this example P(A) = {ϕ, {1}, {2}, A}. The **cardinality** of a set A is the number of elements in A, denoted by |A|.

Theorem: If $|A| = n$ then $|P(A)| = 2^n$ (Every set with n elements has 2^n subsets.)

4.3 Find P(A) and |P(A)| for each set A to verify the above theorem.
- a) $A = \{1, 2, 3\}$
- b) $A = \{1, 3, 5, 7\}$
- c) $A = \phi$
- d) $A = \{\phi, \{1\}\}$

4.4 Prove the above theorem by mathematical induction.

The **cross product** of A and B is the set $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$.

Example: If $A = \{1, 2, 3\}$ and $B = \{x, y\}$ then
$A \times B = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}$
$B \times A = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\}$

Theorem: If $|A| = m$ and $|B| = n$ then $|A \times B| = m \times n$

4.5 Let $A = \{2, 3, 5, 7\}$ and $B = \{1, 2, 4\}$. Evaluate each cardinality.
- a) $|P(A \cup B)|$
- b) $|P(A \cap B)|$
- c) $|P(A - B)|$
- d) $|P(A \oplus B)|$
- e) $|P(A \times B)|$

Theorem: If there are k sets with n elements in all then one of the sets must contain at least $\lceil n/k \rceil$ elements. (The Pigeonhole Principle)

Example: The University has 8 faculties. Given any group of 9 students, at least $\lceil 9/8 \rceil = 2$ of them must belong in the same faculty. And with 42 students at least $\lceil 42/8 \rceil = 6$ must belong in the same faculty

4.6 What is the minimum number of students to ensure the following is true?
- a) 13 of them must be in the same faculty
- b) 2 of them have their birthdays in the same month
- c) 5 of them have their birthdays in the same month
- d) 5 of them have the same birthdays

Theorem: $|A \cup B| = |A| + |B| - |A \cap B|$ (The Inclusion-Exclusion Principle)

Example: How many positive integers ≤ 100 are multiples of 2 or multiples of 3?
Solution: $A = \langle 2 \rangle = \{2 , 4 , 6 , \ldots , 100\}$, $|A| = \lfloor 100/2 \rfloor = 50$
$B = \langle 3 \rangle = \{3 , 6 , 9, \ldots , 99\}$, $|B| = \lfloor 100/3 \rfloor = 33$
$A \cap B = \langle LCM(2,3) \rangle = \langle 6 \rangle = \{6, 12, 18 , \ldots , 96\}$, $|A \cap B| = \lfloor 100/6 \rfloor = 16$
$|A \cup B| = 50 + 33 - 16 = 67$

4.7 How many positive integers ≤ 200 are multiples of
- a) 3 or 5?
- b) 4 or 6?
- c) not multiples of 2 or 17?
- d) not multiples of 12 or 16?

Theorem: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Example: How many positive integers ≤ 100 are multiples of 4, 5, or 6?
Solution: $A = \langle 4 \rangle$, $|A| = \lfloor 100/4 \rfloor = 25$
$B = \langle 5 \rangle$, $|B| = \lfloor 100/5 \rfloor = 20$
$C = \langle 6 \rangle$, $|C| = \lfloor 100/6 \rfloor = 16$

$A \cap B = \langle LCM(4,5) \rangle = \langle 20 \rangle$, $|A \cap B| = \lfloor 100/20 \rfloor = 5$
$A \cap C = \langle LCM(4,6) \rangle = \langle 12 \rangle$, $|A \cap C| = \lfloor 100/12 \rfloor = 8$
$B \cap C = \langle LCM(5,6) \rangle = \langle 30 \rangle$, $|B \cap C| = \lfloor 100/30 \rfloor = 3$
$A \cap B \cap C = \langle LCM(4,5,6) \rangle = \langle 60 \rangle$, $|A \cap B \cap C| = \lfloor 100/60 \rfloor = 1$
$|A \cup B \cup C| = 25 + 20 + 16 - 5 - 8 - 3 + 1 = 46$

4.8 How many positive integers ≤ 1000 are multiples of

a) 2, 3, or 5?
b) 4, 6, or 20?
c) not multiples of 4, 6, or 20?
d) not multiples of 8, 12, or 20?

4.9 Generalize the above theorem for four sets: $|A \cup B \cup C \cup D|$

A **combination** of elements is the set containing those elements, whereas a **permutation** is like a set but with specific ordering of the elements. For example there are 6 different permutations of the elements A, B, C, namely ABC, ACB, BAC, BCA, CAB, and CBA.

Theorem: There are n! different permutations of n elements.

4.10 How many different permutations of the alphabet {A, B, C, … , Z} which

a) contain the word CAR or BYTE?
b) contain the word NO or YES or WHAT?
c) do not contain the word AND or OR or XOR?
d) do not contain the word BY or DNA or COMPUTER?

4.11 A **multiset** is like a set but with repetition of elements allowed. How many different permutations are there of the elements taken from

a) the multiset {A, B, B, C}?
b) the word DISCRETE?
c) the word MATHEMATICS?
d) the word UNUSUAL?

4.12 A **string** over a set Σ is a sequence of elements of Σ. For example over $\Sigma = \{0, 1\}$ the sequence 1011011011101… is a string, whether or not the length is finite. Let Σ^n denote the set of all strings of length n over Σ.

a) Find Σ^3 for $\Sigma = \{0, 1\}$.
b) Find Σ^2 for $\Sigma = \{a, b, c\}$.
c) If $|\Sigma| = m$, what is $|\Sigma^n|$?

C(n, k) denotes the number of subsets which contain k elements from a set with n elements. For example C(3, 2) = 3 because there are 3 subsets of {a, b, c} which have 2 elements, namely {a, b}, {a, c}, and {b, c}.

Theorem: $C(n, k) = \dfrac{n!}{k!\,(n-k)!}$.

Example: If |S| = 10 how many subsets of S are there that contain 8 elements?
Solution: C(10, 8) = 10! / (8! 2!) = (1 2 3 4 5 6 7 8 9 10) / (1 2 3 4 5 6 7 8 1 2) = 45.

4.13 Let |S| = 7. How many subsets does S have which contain

a) 4 elements?
b) 3 elements?
c) 7 elements?
d) more than 5 elements?
e) at least 1 element?

4.14 Evaluate C(n, 0) + C(n, 1) + C(n, 2) + … + C(n, n).

Theorem: There are C(n+k−1, k) non-negative integer solutions of the equation
$x_1 + x_2 + x_3 + \ldots + x_n = k$

Example: How many non-negative integer solutions of a + b + c = 8?
Solution: C(3+8−1, 8) = C(10,8) = 45.

4.15 How many integer solutions of x + y + z = 11 with each given condition?

a) x, y, z are non-negative
b) x, y, z are positive
c) $x \geq 1$, $y \geq 2$, and $z \geq 3$
d) $x \leq 3$, $y \leq 4$, and $z \leq 6$
e) $x \leq 5$, $y \leq 2$, and $z \leq 7$

The **probability** of an event (finite set) A under the assumption that each element is equally likely, is given by p(A) = |A| / |S|, where S is the sample space of all possible events.

Example: A pair of dice is rolled. What is the probability that the sum is 7?
Solution: The sample space is S = {(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),(5,1),(5,2).(5,3),(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)}. The event A = {(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)}. Hence p(A) = 6/36 = 1/6.

4.16 If two dice are rolled, what is the probability of each event below?

a) The sum is 9
b) Two equal numbers
c) At least one 6
d) The sum is at least 9

4.17 In a group of 5 men and 5 women, four people will be chosen. Find the probability of each event given below.

a) All four are women
b) Equal number of men and women
c) At least two men
d) At least one man and one woman

Chapter 5
Binary Relations

Any subset of $A \times A$ is a **binary relation** on the set A.

Examples: The following are some, but not all, binary relations on {1, 2, 3}.

1) $\{(1, 2), (2, 3)\}$
2) $\{(2, 2)\}$
3) $\{(1, 2), (1, 3), (2, 1), (3, 3)\}$
4) ϕ

5.1 If |A| = n how many different relations on A are there?

If $R \subseteq A \times A$ is a relation then the **inverse** $R^{-1} = \{(b, a) \mid (a, b) \in R\}$ is also a relation on A. And if $S \subseteq A \times A$ is another relation then the **composition** of R and S is a relation on A defined by

$$S \circ R = \{(a, c) \mid (a, b) \in R \text{ and } (b, c) \in S\}.$$

Example: Let $A = \{1, 2, 3, 4\}$, $R = \{(1, 2), (2, 4), (3, 1)\}$, and $S = \{(1, 1), (2, 3), (4, 3)\}$. Then
$R^{-1} = \{(2, 1), (4, 2), (1, 3)\}$
$S^{-1} = \{(1, 1), (3, 2), (3, 4)\}$
$S \circ R = \{(1, 3), (2, 3), (3, 1)\}$
$R \circ S = \{(1, 2), (2, 1), (4, 1)\}$

In the case $R \subseteq A \times A$ define $R^2 = R \circ R$ and $R^3 = R \circ R \circ R$, ... , also $R^{-2} = R^{-1} \circ R^{-1}$ etc.

5.2 Let $A = \{1, 2, 3, 4\}$ and $R = \{(1, 2), (2, 1), (2, 4), (3, 1), (3, 4)\} \subseteq A \times A$.
a) Find R^2
b) Find R^3
c) Find R^{-2}
d) Find $(R^2)^{-1}$

Certain properties of a relation $R \subseteq A \times A$ are important:

1) **reflexive**: $(a, a) \in R \ \forall\, a \in A$
2) **symmetric:** $(a, b) \in R \rightarrow (b, a) \in R \ \forall\, a \wedge b \in A$
3) **anti-symmetric:** $(a, b) \in R \rightarrow (b, a) \notin R \ \forall\, a \neq b \in A$
4) **transitive:** $(a, b) \wedge (b, c) \in R \rightarrow (a, c) \in R \ \forall\, a, b, c \in A$

Note that R is symmetric when $R = R^{-1}$ and anti-symmetric when $R \cap R^{-1} = \phi$ or contains only elements of the form (a, a), whereas R is transitive when $R^2 \subseteq R$.

Example: Let $A = \{1, 2, 3\}$ and consider three relations on A:
$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$
$R_2 = \{(1, 1), (1, 3), (2, 2), (3, 2)\}$
$R_3 = \{(1, 2), (1, 3), (2, 3)\}$
For R_1 : reflexive (T) symmetric (T) anti-symmetric (F) transitive (T)
For R_2 : reflexive (F) symmetric (F) anti-symmetric (T) transitive (F)
For R_3 : reflexive (F) symmetric (F) anti-symmetric (T) transitive (T)

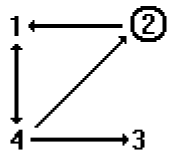
5.3 Let $A = \{1, 2, 3, 4\}$. Find the truth values of the four propositions for each $R \subseteq A \times A$.
a) $R = \{(a, b) \mid a \leq b\}$
b) $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$
c) $R = \{(1, 1), (1, 3), (2, 1), (2, 2), (2, 4)\}$
d) $R = \{(a, b) \mid a + b \geq 5\}$

5.4 Let A = {1, 2, 3, 4}. Give any example of a relation $R \subseteq A \times A$ which is

a) reflexive (T) symmetric (T) anti-symmetric (F) transitive (F)
b) reflexive (F) symmetric (T) anti-symmetric (F) transitive (F)
c) reflexive (F) symmetric (T) anti-symmetric (F) transitive (T)
d) reflexive (F) symmetric (F) anti-symmetric (F) transitive (F)
e) reflexive (T) symmetric (T) anti-symmetric (T) transitive (T)

A relation $R \subseteq A \times A$ can be represented by a **digraph** in which each element of A is represented by a **vertex** and each element (a, b) of R is represented by an **edge** with direction from a to b. In the case a = b the edge is a **loop**.

Example: A = {1, 2, 3, 4} and R = {(1, 4), (2, 1), (2, 2), (4, 1), (4, 2), (4, 3)}.



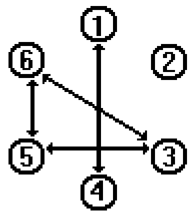
5.5 Draw the digraph for each of the relations in Problem 5.3.

5.6 What characterizes the digraph of a relation with each of the following properties?

a) reflexive
b) **anti-reflexive** [meaning that $(a, a) \notin R \ \forall \ a \in A$]
c) symmetric
d) anti-symmetric
e) transitive

$R \subseteq A \times A$ is an **equivalence relation** if it is reflexive, symmetric, and transitive. If R is an equivalence relation then A is partitioned into subsets such that in each subset every two vertices are connected by an edge. These subsets are the **equivalence classes** of A under the relation R.

Example: The following digraph shows that R is an equivalence relation. (Why?)
The equivalence classes are {1, 4}, {2}, and {3, 5, 6}. (Why?)



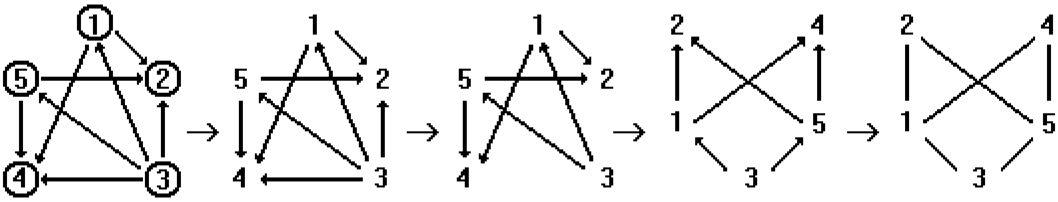
5.7 Prove that R is an equivalence relation and then find the equivalence classes.

a) A = {0, 1, 2, 3, 4, 5, 6} and R = {(a, b) | a + b is even}
b) A = {1, 2, 3, 4} and R = {(a, b) | a = b}
c) A = {0, 5, 8, 9, 10, 11} and R = {(a, b) | a − b is a multiple of 3}
d) A = {1, 2, 3, 6, 7, 11} and R ={(a, b) | a mod 5 = b mod 5}
e) A = {1, 9, 21, 44, 50, 99, 101} and R ={(a, b) | (a − b) mod 10 = 0}

$R \subseteq A \times A$ is a **partial order** relation if it is reflexive, anti-symmetric, and transitive. In this case the digraph of R can be simplified into a **Hasse diagram** after these 4 steps:

1) Do not draw loops.
2) Do not draw (a, c) whenever there are (a, b) and (b, c).
3) Redraw the remaining graph so that all edges point upward.
4) Do not draw the directions.

Example: The following digraph shows that R is a partial order relation. (Why?)
The 4 steps above lead to the Hasse diagram of R.



5.8 Prove that R is a partial order relation and then draw the Hasse diagram.

a) A = {1, 2, 3, 4} and R = {(a, b) | a $\leq$ b}
b) A = {1, 2, 6, 12, 24} and R = {(a, b) | a is a divisor of b}
c) A = {1, 2, 6, 10, 20, 30} and R = {(a, b) | b mod a = 0}
d) A = {1, 5, 7, 10, 35, 70} and R = {(a, b) | b mod a = 0}

$R \subseteq A \times A$ is a **total order** relation if it is a partial order relation in which every two vertices are connected by an edge. The partial order relation in the previous example is not a total order because there is no edge between 2 and 4. Moreover the Hasse diagram of a total order relation can always be drawn as a vertical line.

5.9 Which ones of the partial order relations in Problem 5.8 are total order?

If A = {1, 2, 3, … , n} then a binary relation $R \subseteq A \times A$ can be represented by a **zero-one matrix** (m_{ij}) of size n×n where m_{ij} = 0 if (i, j) $\notin$ R and m_{ij} = 1 if (i, j) $\in$ R.

Example: A = {1, 2, 3}. Find the zero-one matrix of R = {(1,1), (1,3), (2,1), (3,2), (3,3)}.

Solution: $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

5.10 Represent the relations given in Problem 5.3 using zero-one matrices.

5.11 Convert these zero-one matrices to digraphs.

a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ c) $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ d) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

The **transitive closure** of $R \subseteq A \times A$ is the smallest transitive relation containing R.

Theorem: The transitive closure of R is given by $R \cup R^2 \cup R^3 \cup \ldots \cup R^n$ where n = |A|.

5.12 Let A = {1, 2, 3, 4}. Use the above theorem to find the transitive closure of $R \subseteq A \times A$.

a) R = {(1, 2), (2, 1), (2, 3), (3, 4)}
b) R = {(1, 1), (1, 2), (2, 1), (4, 3)}
c) R = {(1, 1), (1, 4), (2, 1), (2, 2), (3, 3), (4, 4)}
d) R = {(1, 4), (2, 1), (2, 4), (3, 2), (3, 4), (4, 3)}

5.13 Find the zero-one matrix of the transitive closure for each relation in Problem 5.11.

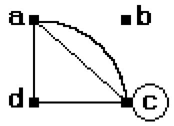
5.14 Given R and its zero-one matrix M, discover a way to compute M^2, the corresponding matrix of R^2, then redo Problem 5.13 using only matrices.

5.15 Discuss the obvious definitions of **reflexive closure** and **symmetric closure** and how we might find them.

Chapter 6
Graph Theory

A **graph** consists of two components: a set of vertices and a multiset of edges and loops.

Example: $G = \{a, b, c, d\} \cup \{ac, ac, ad, cc, cd\}$ which we represent graphically as follows.



There are two matrical representations of a graph G with m vertices and n edges:

1) The **adjacency matrix** of G is an $m \times m$ matrix defined by m_{ij} = the number of edges between vertex i and vertex j.
2) The **incidence matrix** of G is an $m \times n$ matrix defined by $m_{ij} = 1$ if edge j is incidence on vertex i and $m_{ij} = 0$ otherwise.

Example: The adjacency matrix and the incidence matrix of G above are respectively

$$\begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

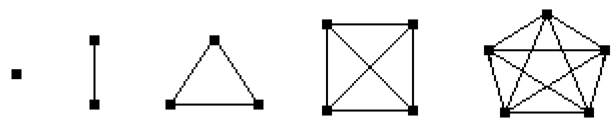
6.1 Convert these adjacency matrices to incidence matrices.

a) $\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

6.2 Convert these incidence matrices to adjacency matrices.

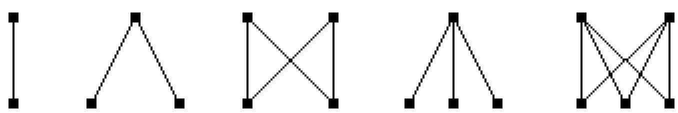
a) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$

A graph is **simple** if it has neither loops nor multiple edges. A simple graph in which every two vertices are connected by an edge is called a **complete** graph. Let K_n denote the complete graph with n vertices, as pictured below for n = 1, 2, 3, 4, 5.



K_1 K_2 K_3 K_4 K_5

A **complete bipartite** graph is a simple graph whose vertices can be partitioned into two subsets such that two vertices are connected if and only if they belong to different subsets. Let $K_{m,n}$ denote the complete bipartite graph with the partition into m and n vertices:



$K_{1,1}$ $K_{1,2}$ $K_{2,2}$ $K_{1,3}$ $K_{2,3}$

6.3 Write the adjacency matrix and the incidence matrix for K_4 and for $K_{3,1}$ and $K_{2,2}$.

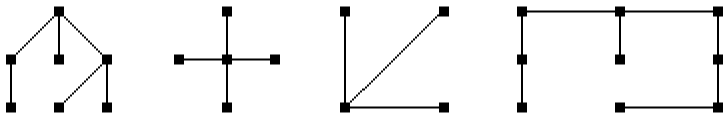
The **degree** of a vertex is the number of edges incidence on it, where a loop counts as two edges. The **degree** of a graph is the sum of all the degrees of its vertices. In the example G = {a, b, c, d} $\cup$ {ac, ac, ad, cc, cd} given earlier, the degree of G = deg(a) + deg(b) + deg(c) + deg(d) = 3 + 0 + 5 + 2 = 10.

Theorem: The degree of any graph is twice its number of edges.

6.4 Find the formula for the number of edges and the degree of K_n and for $K_{m,n}$.

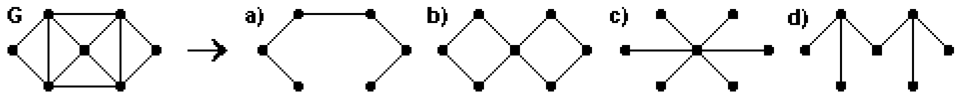
A **tree** is a connected graph whose number of edges is one less than the number of vertices.

Examples:



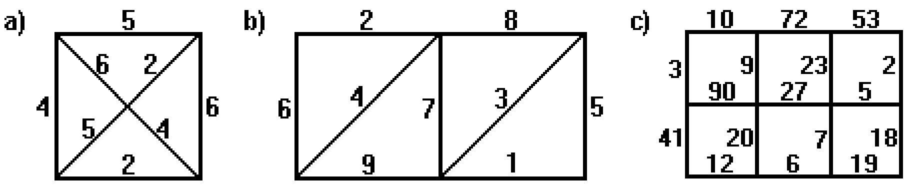
6.5 For which n is K_n a tree? How about $K_{m,n}$?

A **spanning tree** of a graph G is a tree subgraph of G containing all the vertices of G. In the example below (a) is not a spanning tree of G because it lacks one vertex of G, (b) is not because it is not a tree, (c) is not because it is not a subgraph of G, (d) is a spanning tree of G.



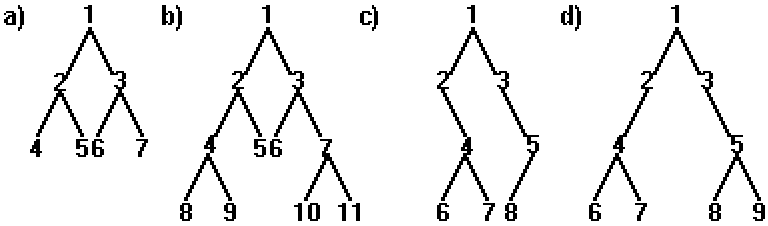
A graph is **weighted** if each edge is associated with a numerical value. A **minimal spanning tree** of a weighted graph is one with smallest possible total value. One way to obtain a minimal spanning tree from a weighted graph is by repeatedly removing the edge with largest value, provided that this action does not disconnect the graph, until what is left forms a tree.

6.6 Find a minimal spanning tree for each weighted graph below.



A vertex in a tree can be selected as the **root**, which is placed topmost and from which every edge is directed downward. In this case a vertex one-edge down is called a **child** of the one above it. A **labeled binary tree** is a rooted tree in which every vertex has at most two children which are distinguished as a left and/or a right child, if any.

Examples:



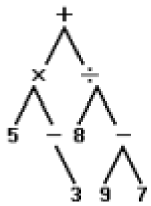
There are 3 common algorithms for traversing the vertices of a labeled binary tree:

1) **pre-order** traversal: ROOT → LEFT → RIGHT
2) **post-order** traversal: LEFT → RIGHT → ROOT
3) **in-order** traversal: LEFT → ROOT → RIGHT

Example: Apply these algorithms to the labeled binary tree (a) in the above example.
pre-order: 1, 2, 4, 5, 3, 6, 7
post-order: 4, 5, 2, 6, 7, 3, 1
in-order: 4, 2, 5, 1, 6, 3, 7

6.7 Complete the example for (b), (c), and (d) using the 3 algorithms.

Labeled binary trees can be used to represent mathematical expressions in accordance with the in-order traversal. For example $[5 \times (-3)] + [8 \div (9 - 7)]$:

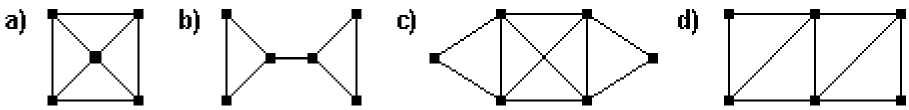


6.8 Represent these expressions using labeled binary trees.

a) $(x \times y) + [(y \div x) - (x + y)^3]$
b) $(A \cup B) \oplus [(A \cap C) \cup (B - C)]$
c) $(p \to \neg q) \leftrightarrow [\neg p \wedge (q \oplus r)]$

An **Euler** path in a graph is a continuous walk through all its edges without repetition. If the walk ends at the same starting vertex, we call it an Euler circuit.

6.9 Are these graphs Euler paths/circuits?

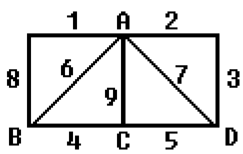


Theorem: A connected graph is an Euler circuit if and only if the degree of each vertex is even. Otherwise it is an Euler path if and only if exactly two vertices have odd degrees.

6.10 For which n is K_n an Euler path/circuit? How about $K_{m,n}$?

The **Chinese postman problem** asks for a circuit in a weighted graph that has a least weight. If there is, of course, an Euler circuit would be an ideal solution; else the walk would necessarily repeat some edges.

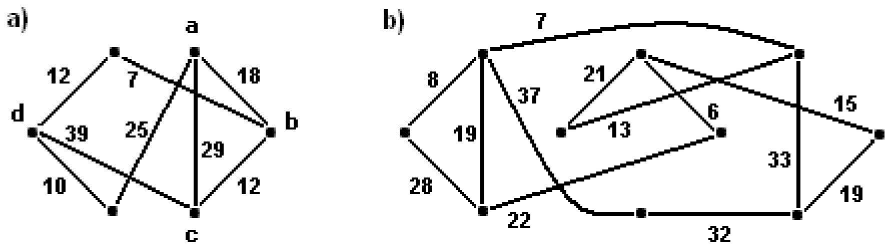
Example: Solve the Chinese postman problem for the following graph.



Solution: There are four vertices of odd degree, which we label A,B,C,D, above. If we build two extra edges to connect them in pairs, the new graph would be Euler circuit. Each extra edge is actually a walk through the existing edges, so we study all the possibilities of pairing up the four vertices as follows.
{A,B} + {C,D} = 6 + 5 = 11
{A,C} + {B,D} = 9 + (4 + 5) = 18

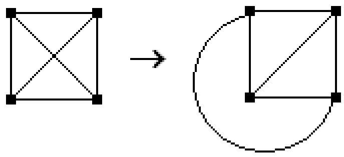
{A,D} + {B,C} = (2 + 3) + 4 = 9
The minimal solution involves walking through all the edges (of weight 45) plus the (least cost) repetition from A to D (of weight 5) and from B to C (of weight 4). The total cost will be 45 + 9 = 54.

6.11 Solve the Chinese postman problems for the weighted graphs below.



A graph is **planar** if it can be drawn without crossing any edge.

Example: K_4 is planar.

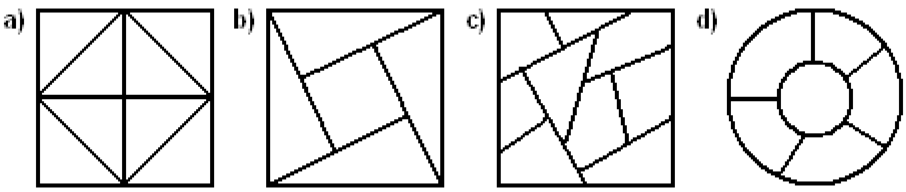


6.12 Are these graphs planar ?

a) K_5
b) $K_{2,2}$
c) $K_{2,3}$
d) $K_{3,3}$

This particular drawing of a planar graph is called a **map**, and it partitions the plane into a number of regions. For example the map of K_4 partitions the plane into 3 interior regions. The **chromatic number** of a map is the minimum number of colors needed to color the interior regions of the map such that regions which share an edge are of different colors.

6.13 Find the chromatic numbers of these maps.



Theorem: The chromatic number of any map is at most 4. (The Four-Color Theorem)

6.14 Draw a map with 4 interior regions and with chromatic number equals 4.

The **dual graph** G of a map M is defined as follow.

1) The vertices of G are the interior regions of M
2) The edges of G are the boundaries between two regions of M

Example: The dual graph of K_4 is K_3 (Verify it!)

6.15 Draw the dual graphs for the maps given in Problem 6.13, then find their chromatic numbers again by coloring the vertices of the dual graphs!

Appendix 1
Personalized Projects

1. Convert your university number to (a) binary (b) hexadecimal and (c) octal.

2. Use the Euclidean algorithm to compute GCD (m, n) where m is your university number and n is the same number with the digits reversed from right to left.

3. How many different permutations can be formed using all the digits in your university number?

4. The set A consists of the digits in your university number and R = {(a, b) | a mod 3 = b mod 3}. Show that R is an equivalence relation and then find the equivalence classes.

5. The set A consists of the digits in your university number and R = {(a, b) | b mod a = 0}. Show that $R \cup \{(0, 0)\}$ is a partial order relation and then draw the Hasse diagram.

6. Write your university number in binary and then enter the digits into a 5×5 zero-one matrix, starting from the upper left corner. Ignore any leftover digits.
    a) Find the elements of R and draw its digraph.
    b) Is R reflexive, symmetric, anti-symmetric, or transitive?
    c) Find the matrix for the transitive closure of R.

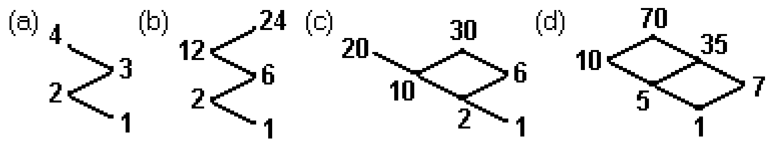
Appendix 2
Selected Answers

1.1 (a) Amman is in Jordan and $2 + 2 \neq 5$ (b) T (c) Amman is not in Jordan or $2 + 2 \neq 5$ (d) T
1.2 (a) $q \leftrightarrow \neg r$ (b) $p \oplus \neg p$ (c) $\neg r \rightarrow \neg q$ (d) $\neg(p \vee r)$ (e) $r \rightarrow (q \vee p)$
1.3 (a) F T T T (b) T T F F (c) T T F T (d) T F T T T F T F (e) F F T F F F F F
1.6 (a) If I do not get good mark then I do not study hard (b) If it is hot then it does not rain
1.7 (a) tautology (b) tautology (c) contingency (d) tautology (e) contradiction
1.8 (a) valid (b) invalid
1.10 (a) F (b) T (c) F (d) T
1.11 (a) $(p \vee \neg q) \wedge (p \vee q) \equiv (p \wedge q) \vee (p \wedge \neg q)$ (b) $(p \vee \neg q) \equiv (p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$
1.12 (a) $(\neg p \vee q) \wedge (p \vee q)$ (b) $(p \wedge \neg q)$ (c) $(\neg p\vee\neg q\vee\neg r) \wedge (\neg p\vee\neg q\vee r) \wedge (\neg p\vee q\vee\neg r) \wedge (p\vee\neg q\vee\neg r)$

2.4 (a) F (b) F (c) $0 < x < 1$
2.5 (a) T (b) T (c) F (d) $(x, 0) \vee (0, y)$
2.6 (a) T (b) F
2.7 (a) T (b) F (c) T (d) T (e) T

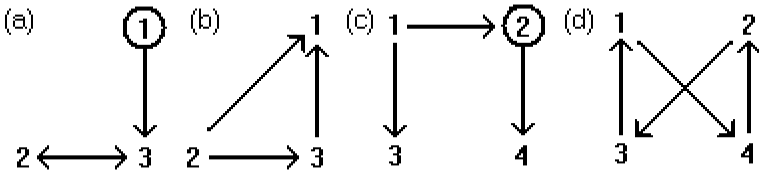
3.1 (a) 42 (b) 328 (c) 183 (d) 65
3.2 (a) 100101 (b) 1100011 (c) 111110100 (d) 1111100111
3.3 (a) 1534 (b) 3021 (c) 41120 (d) 69905
3.4 (a) 25 (b) 63 (c) 1F4 (d) 3E7
3.5 (a) 10111111110 (b) 101111001101 (c) 1010000010100000 (d) 10001000100010001
3.6 (a) 2A (b) 148 (c) B7 (d) 41
3.7 (a) 45 (b) 143 (c) 764 (d) 1747
3.8 (a) 0.8125 (b) 0.015625 (c) 7.875 (d) 0.65625 (e) 273.06665039062
3.9 (a) 0.00001 = 0.08 (b) 0.110001 = 0.C4 (c) 0.101 = 0.A (d) 0.010101… = 0.555…
3.10 (a) 0 (b) 5 (c) 3 (d) 11
3.13 (a) 5 (b) 3 (c) 24 (d) 25
3.14 (a) 6325 (b) 40959 (c) 2736 (d) 29000
3.15 (a) $f(n) = f(n-1) + f(n-2) + f(n-3)$ (b) $f(n-1) + 10$ (c) $n \times f(n-1)$ (d) $f(n-1) + \lfloor ½ f(n-2) \rfloor$
3.16 (a) $f(n) = 3(2^n) - 2(-1)^n$ (b) $f(n) = 2^n + n(2^{n-1})$ (c) $a_n = ¼ [3^{n+1} + (-1)^n]$ (d) $a_n = 3n + 1$
3.17 (a) $f(n) = \{(1+\sqrt{5})/2\}^n / \sqrt{5} - \{(1-\sqrt{5})/2\}^n / \sqrt{5}$

4.1 (a) {2,4} (b) {0,6} (c) {0,1,3,5,6} (d) {1,2,3,4,5}
4.2 (a) T (b) T (c) F (d) T (e) T
4.3 (a) $|\{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}|=8$ (c) $|\{\phi\}|=1$ (d) $|\{\phi, \{\phi\}, \{\{1\}\}, \{\phi, \{1\}\}|=4$
4.5 (a) 64 (b) 2 (c) 8 (d) 32 (e) 4096
4.6 (a) 97 (b) 13 (c) 49 (d) 1465
4.7 (a) 93 (b) 67 (c) 94 (d) 176
4.8 (a) 734 (b) 333 (c) 667 (d) 816
4.10 (a) 24! + 23! – 21! (b) 25! + 24! – 22! – 21! + 20! (c) 26! – 25! – 24! + 23!
4.11 (a) 12 (b) 20160 (c) 4989600 (d) 840
4.12 (a) {000, 001, 010, 011, 100, 101, 110, 111} (b) {aa, ab, ac, ba, bb, bc, ca, cb, cc}
4.13 (a) 35 (b) 35 (c) 1 (d) 8 (e) 127
4.15 (a) 78 (b) 45 (c) 21 (d) 6 (e) 9
4.16 (a) 1/9 (b) 1/6 (c) 11/36 (d) 5/18
4.17 (a) 1/42 (b) 10/21 (c) 31/42 (d) 20/21

5.2 (a) {(1,1),(1,4),(2,2),(3,2)} (b) {(1,2),(2,1),(2,4),(3,1),(3,4)} (c&d) {(1,1),(2,2),(2,3),(4,1)}
5.3 (a&b) reflexive, anti-symmetric, transitive (c) anti-symmetric (d) symmetric
5.7 (a) {0,2,4,6}, {1,3,5} (b) {1}, {2}, {3}, {4} (c) {0,9}, {5,8,11}, {10} (d) {1,6,11}, {2,7}, {3}
5.8


5.9 (a) T (b) T (c) F (d) F

5.10 (a) $\begin{bmatrix}1&1&1&1\\0&1&1&1\\0&0&1&1\\0&0&0&1\end{bmatrix}$ (b) $\begin{bmatrix}1&1&1&1\\0&1&0&1\\0&0&1&0\\0&0&0&1\end{bmatrix}$ (c) $\begin{bmatrix}1&0&1&0\\1&1&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}$ (d) $\begin{bmatrix}0&0&0&1\\0&0&1&1\\0&1&1&1\\1&1&1&1\end{bmatrix}$

5.11


5.12 (a) {(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,4)} (b) R $\cup$ (2,2) (c) R $\cup$ (2,4) (d) A $\times$ A

5.13 (a) $\begin{bmatrix}1&1&1\\0&1&1\\0&1&1\end{bmatrix}$ (b) $\begin{bmatrix}0&0&0\\1&0&1\\1&0&0\end{bmatrix}$ (c) $\begin{bmatrix}0&1&1&1\\0&1&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix}$ (d) $\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{bmatrix}$

6.1 (a) $\begin{bmatrix}1&1&1\\0&1&1\end{bmatrix}$ (b) $\begin{bmatrix}1&1&1&1&0&0\\0&0&0&0&1&0\\0&1&1&1&0&1\end{bmatrix}$ (c) $\begin{bmatrix}1&1&0&0&0&0&0\\1&0&1&1&1&0&0\\0&1&1&1&0&1&1\\0&0&0&0&1&0&0\end{bmatrix}$

6.2 (a) $\begin{bmatrix}0&1&0\\1&1&2\\0&2&1\end{bmatrix}$ (b) $\begin{bmatrix}0&1&0&1\\1&0&1&1\\0&1&1&1\\1&1&1&0\end{bmatrix}$ (c) $\begin{bmatrix}0&0&2&0\\0&1&0&1\\2&0&2&1\\0&1&1&0\end{bmatrix}$

6.4 (a) deg Kn = n(n – 1) (b) deg Km,n = 2mn
6.5 (a) n=1 or n=2 (b) m=1 or n=1
6.6 (a) 12 (b) 17 (c) 115
6.7 (b) pre (1 2 4 8 9 5 3 6 7 10 11) post (8 9 4 5 2 6 10 11 7 3 1) in (8 4 9 2 5 1 6 3 10 7 11)
6.7 (c) (12467358), (67428531), (26471385) (d) (124673589), (674289531), (647213859)
6.9 (a) F (b) Euler path (c) Euler circuit (d) Euler path
6.11 (a) 199 (b) 220
6.12 (a) F (b) T (c) T (d) F
6.13 (a) 2 (b) 3 (c) 4 (d) 4

Homework 1

1. Convert the binary number to decimal.
   (a) 10001
   (b) 101010
   (c) 1101110
   (d) 101100111
2. Convert the decimal number to binary.
   (a) 11
   (b) 110
   (c) 2017
   (d) 4096

p	q	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$	$p \oplus q$
1	1	1	1	1	1	0
1	0	0	1	0	0	1
0	1	0	1	1	0	1
0	0	0	0	1	1	0

3. Draw the truth table for each proposition.
   (a) $\neg p \vee \neg q$
   (b) $\neg q \wedge (p \wedge q)$
   (c) $(p \vee \neg q) \wedge (\neg p \vee q)$
   (d) $(p \wedge q) \vee r$
4. Draw the truth table for each proposition.
   (a) $p \rightarrow (q \rightarrow r)$
   (b) $\neg(p \leftrightarrow q) \rightarrow q$
   (c) $(p \oplus \neg q) \vee (\neg p \leftrightarrow q)$
   (d) $\{(p \wedge q) \rightarrow r\} \oplus \{\neg p \vee (q \leftrightarrow r)\}$
5. Use truth table to prove the equivalence.
   (a) $\neg(p \vee q) \equiv \neg p \wedge \neg q$
   (b) $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
   (c) $p \rightarrow (q \rightarrow r) \equiv q \rightarrow (p \rightarrow r)$
   (d) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
6. Prove true or false.

(a) $p \to \neg q \equiv q \to \neg p$
(b) $\neg(p \oplus q) \equiv \neg p \leftrightarrow \neg q$
(c) $p \to (q \to r) \equiv (p \to q) \to r$
(d) $p \vee (q \oplus r) \equiv (p \vee q) \oplus (p \vee r)$

7. Convert each proposition to a CNF and to a DNF.

   (a) $\neg(p \wedge q) \to p$
   (b) $(p \oplus \neg q) \leftrightarrow (\neg p \vee q)$
   (c) $(p \to q) \to r$
   (d) $\{(p \wedge q) \to r\} \oplus \{\neg p \vee (q \leftrightarrow r)\}$

8. Convert each CNF to a DNF and each DNF to a CNF.

   (a) $(p \wedge \neg q) \vee (\neg p \wedge q)$
   (b) $(p \vee \neg q) \wedge (\neg p \vee \neg q)$
   (c) $(p \vee q \vee r) \wedge (p \vee \neg q \vee r) \wedge (p \vee q \vee \neg r)$
   (d) $(p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge \neg r)$

9. Evaluate the set operations with $A = \{1, 2, 3, 4, 5\}, B = \{2, 4, 6\}, C = \{1, 3, 5\}$, and $D = \{2, 5, 7, 9\}$.

   (a) $(A \cap B) \cup (C \cap D)$
   (b) $(B - A) \cap (C \cup D)$
   (c) $\{A - (C - D)\} - B$
   (d) $(A \oplus B) \oplus (C \oplus D)$

10. Use Venn diagram to simplify each expression.

    (a) $(A \cap B) \oplus (A - B)$
    (b) $\{A - (A - B)\} \oplus B$
    (c) $(A \cup B) \oplus (A \cap B)$
    (d) $(A \cup B) \oplus (A \oplus B)$

11. Find the elements of $P(A)$.

    (a) $A = \{a, b, c\}$
    (b) $A = \{2, 3, 4, 5\}$
    (c) $A = \{x, \{7\}\}$
    (d) $A = \emptyset$

    $\boxed{|A| = n \to |P(A)| = 2^n}$

12. Evaluate the cardinality with $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$.

    (a) $|P(A \cup B)|$
    (b) $|P(A - B)|$
    (c) $|P(A) - P(B)|$
    (d) $|P(P(A \oplus B))|$

Homework 2

1. Evaluate the floor function.

   (a) $\lfloor 3.999 \rfloor$

   (b) $\lfloor \sqrt{450} \rfloor$

   (c) $\lfloor 100/7 \rfloor$

   (d) $\lfloor -111/22 \rfloor$

   $\boxed{m \bmod n = m - \lfloor \frac{m}{n} \rfloor \times n}$

2. Evaluate the mod operation.

   (a) $11 \bmod 27$

   (b) $100 \bmod 7$

   (c) $-111 \bmod 22$

   (d) $12345 \bmod 15$

3. Use SSA to evaluate the power mod.

   (a) $2^{33} \bmod 9$

   (b) $3^{63} \bmod 10$

   (c) $5^{511} \bmod 11$

   (d) $11^{100} \bmod 100$

   $\boxed{\gcd(m, n) = \gcd(n, m \bmod n)}$

4. Evaluate the GCD.

   (a) $\gcd(549, 81)$

   (b) $\gcd(1234, 5678)$

   (c) $\gcd(234, 60970)$

   (d) $\gcd(12345, 54321)$

   $\boxed{\operatorname{lcm}(m, n) = \frac{m \times n}{\gcd(m,n)}}$

5. Evaluate the LCM.

   (a) $\operatorname{lcm}(5, 72)$

   (b) $\operatorname{lcm}(12, 18)$

   (c) $\operatorname{lcm}(136, 17)$

   (d) $\operatorname{lcm}(549, 81)$

6. Find integers a and b such that $\gcd(m, n) = am + bn$.

   (a) $m = 27, n = 25$

   (b) $m = 549, n = 81$

(c) $m = 345, n = 215$

(d) $m = 843, n = 2890$

7. Find $a^{-1} \bmod n$.

(a) $7^{-1} \bmod 11$

(b) $5^{-1} \bmod 17$

(c) $5^{-1} \bmod 18$

(d) $13^{-1} \bmod 100$

8. Determine true or false.

(a) $5^{-1} \bmod 7 = 3$

(b) $7^{-1} \bmod 16 = 7$

(c) $10^{-1} \bmod 17 = 12$

(d) $13^{-1} \bmod 20 = 13$

9. Given that $n = p \times q$ in this RSA example, use the encryption key e to compute the decryption key $d := e^{-1} \bmod (p-1)(q-1)$.

(a) $n = 55, e = 3, m = 47$

(b) $n = 65, e = 5, m = 7$

(c) $n = 91, e = 11, m = 5$

(d) $n = 391, e = 7, m = 29$

10. In Problem 9, (i) compute $s := m^e \bmod n$ and (ii) prove that $s^d \bmod n = m$.

$\boxed{|A \cup B| = |A| + |B| - |A \cap B|}$

11. From 1 to 300, count how many:

(a) multiples of 8 or 12

(b) multiples of 8 and 12

(c) not multiples of 12 or 16

(d) multiples of 12 but not 16

$\boxed{|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|}$

12. From 1 to 1000, count how many:

(a) multiples of 6 or 15 or 20

(b) multiples of 12 or 18 or 24

(c) multiples of 8 and 12 and 20

(d) not multiples of 14 or 21 or 30

Homework 3

$\boxed{C(n,k) = \frac{n!}{k! \times (n-k)!}}$

1. Evaluate the binomial coefficients.
   (a) $C(26, 23)$
   (b) $C(25, 22) + C(25, 23)$
   (c) $C(3, 0) + C(3, 1) + C(3, 2) + C(3, 3)$
   (d) $C(4, 0) + C(4, 1) + C(4, 2) + C(4, 3) + C(4, 4)$

2. Count how many subsets of A with the given condition.
   (a) $|A| = 12$; subsets with 7 elements
   (b) $|A| = 12$; subsets with 7 or 8 elements
   (c) $|A| = 20$; subsets with at least 18 elements
   (d) $|A| = 20$; subsets with at least 3 elements

3. Count how many non-negative integer solutions for $A + B + C + D = 18$ such that
   (a) $A \geq 9$
   (b) $A \geq 4$ and $B \geq 7$
   (c) $A \geq 4$ or $B \geq 7$
   (d) $A \geq 4$ or $B \geq 7$ or $C \geq 5$

4. Count how many non-negative integer solutions for $A + B + C = 12$ such that
   (a) $A \leq 5$
   (b) $A \leq 1$ and $B \leq 1$
   (c) $A \leq 1$ or $B \leq 2$ or $C \leq 3$
   (d) $A \leq 4$ and $B \leq 5$ and $C \leq 6$

5. Given the function $S(n)$, find the first 6 terms in the sequence.
   (a) $S(n) = \lfloor \frac{n}{2} \rfloor$
   (b) $S(n) = 2^n - 3$
   (c) $S(n) = n^2 - n$
   (d) $S(n) = n \bmod 3$

6. Find a suitable function $S(n)$ for the given sequence.
   (a) 2, 3, 2, 3, 2, 3, 2, ...
   (b) 1, 2, 5, 10, 17, 26, 37, ...
   (c) 3, 8, 13, 18, 23, 28, 33, ...
   (d) 2, 4, 8, 16, 32, 64, 128, ...

7. Given $S(0) = 1$ and $S(1) = 2$, find $S(5)$ using the recurrence relation.

   (a) $S(n) = S(n-1) + 2S(n-2)$ linear, second-order, homogeneous
   (b) $S(n) = S(n-1) - S(n-2) + 2$ linear, second-order, non-homogeneous
   (c) $S(n) = S(n-1) + S(n-2)^2$ quadratic, second-order, non-homogeneous
   (d) $S(2) = 3, S(n) = S(n-1) + S(n-2) + S(n-3)$ linear, third, homogeneous

8. Find a suitable linear second-order recurrence relation for the given sequence.

   (a) 2, 1, 3, 4, 7, 11, 18, ...
   (b) 1, 2, 4, 7, 12, 20, 33, ...
   (c) 0, 1, 2, 5, 12, 29, 70, ...
   (d) 1, 2, 2, 4, 8, 32, 256, ...

$$\boxed{\begin{array}{l} S(n) = A\,S(n-1) + B\,S(n-2) \to x^2 - Ax - B = 0 \\ x_1 \neq x_2 \to S(n) = C\,(x_1)^n + D\,(x_2)^n \\ x_1 = x_2 \to S(n) = C\,(x)^n + D\,n\,(x)^n \end{array}}$$

9. Find the function $S(n)$ given by its recurrence relation.

   (a) $S(0) = 1, S(1) = 3, S(n) = S(n-1) + 6S(n-2)$
   (b) $S(0) = 2, S(1) = 3, S(n) = S(n-1) + 12S(n-2)$
   (c) $S(0) = 2, S(1) = 3, S(n) = 6S(n-1) - 9S(n-2)$
   (d) $S(0) = 0, S(1) = 1, S(n) = -2S(n-1) + 15S(n-2)$

10. Find a suitable linear second-order homogeneous recurrence relation for the given sequence, and then find the explicit formula for the function $S(n)$.

    (a) 0, 1, 1, 2, 3, 5, 8, ...
    (b) 0, 1, 1, 3, 5, 11, 21, ...
    (c) 2, 1, 3, 4, 7, 11, 18, ...
    (d) 0, 1, 2, 5, 12, 29, 70, ...

11. Prove the formula for all $n \geq 1$ using mathematical induction.

    (a) $2 + 4 + 6 + 8 + \cdots + 2n = n(n+1)$
    (b) $1 + 5 + 25 + 125 + \cdots + 5^n = \frac{5^{n+1}-1}{4}$
    (c) $1 + 4 + 9 + 16 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$
    (d) $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$
    (e) $1(1\,!) + 2(2\,!) + 3(3\,!) + \cdots + n(n\,!) = (n+1)\,! - 1$

12. Prove the inequality using mathematical induction.

    (a) $n < 2^n$ for all $n \geq 1$
    (b) $2^n > n^2$ for all $n \geq 5$
    (c) $2^n < n\,!$ for all $n \geq 4$
    (d) $n\,! > 3^n$ for all $n \geq 7$
    (e) $n\,! < n^n$ for all $n \geq 2$

Homework 4

1. Use digraph to represent the relation R on $A = \{1, 2, 3, 4, 5\}$.

   (a) $R = \{(x, y) \mid x < y\}$

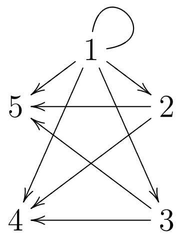
   (b) $R = \{(x, y) \mid x + y \geq 7\}$

   (c) $R = \{(x, y) \mid x - y = 1\}$

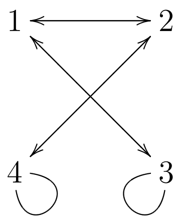
   (d) $R = \{(x, y) \mid (x - y)^2 = 1\}$

2. Convert the digraph to matrix.

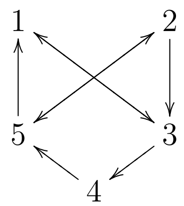
   (a)

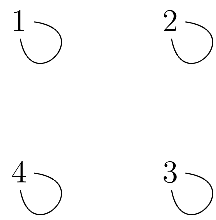
   (b)

   (c)

   (d)

3. Use matrix to represent the relation R on $A = \{1, 2, 3, 4\}$.

   (a) $R = \{(x, y) \mid x \bmod y = 0\}$

   (b) $R = \{(x, y) \mid x \bmod y \neq 0\}$

   (c) $R = \{(x, y) \mid y \bmod x = 0\}$

   (d) $R = \{(x, y) \mid x \bmod y = 1\}$

4. Convert the relation matrix to digraph.

   (a) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

$\boxed{S \circ R = \{(a, c) \mid (a, b) \in R \wedge (b, c) \in S\}}$

5. Given $R = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ and $S = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ find the composition matrix.

   (a) $R \circ S$

   (b) $S \circ R$

   (c) $R \circ S \circ S$

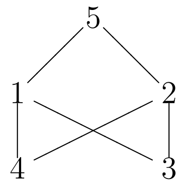
   (d) $S \circ R \circ R$

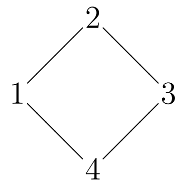
$\boxed{\overline{R} = R \cup R^2 \cup R^3 \cup \cdots \cup R^n}$

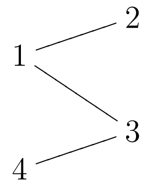
6. Given the relation matrix R, find the matrix for the transitive closure $\overline{R}$.

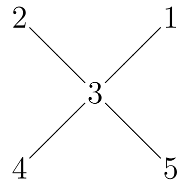
(a) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

7. Convert the Hasse diagram to matrix.

(a) 

(b) 

(c) 

(d) 

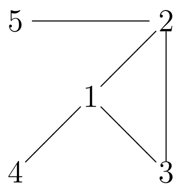
8. Convert the partial order relation matrix to Hasse diagram.

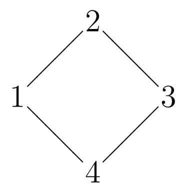
(a) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

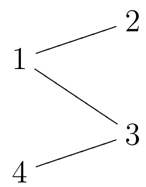
9. Draw the Hasse diagram for the partial order $R = \{(x, y) \mid y \bmod x = 0\}$ on A.

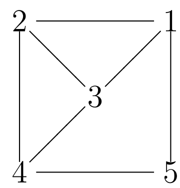
(a) $A = \{1, 2, 4, 8, 16\}$
(b) $A = \{2, 3, 6, 12, 18\}$
(c) $A = \{1, 2, 3, 4, 6, 12\}$
(d) $A = \{2, 3, 12, 18, 36\}$

10. Use incidence matrix to represent the graph.

(a) 

(b) 

(c) 

(d) 

11. Convert the incidence matrix to adjacency matrix.

(a) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

12. Convert the adjacency matrix to incidence matrix.

(a) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

Homework 5

1. For each graph, give the (i) adjacency matrix (ii) incidence matrix.

   (a) K_4

   (b) K_5

   (c) $K_{2,2}$

   (d) $K_{1,3}$

2. Find the degree of each graph.

   (a) K_{10}

   (b) $K_{4,5}$

   (c) The graphs (a), (b), (c), (d) given by the incidence matrix in Problem 11 of Homework 4.

   (d) The graphs (a), (b), (c), (d) given by the adjacency matrix in Problem 12 of Homework 4.

3. Determine which graph is a tree.

   (a) The graphs (a), (b), (c), (d) in Problem 1.

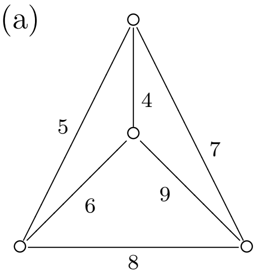
   (b) The graphs (a), (b), (c), (d) in Problem 10 of Homework 4.

   (c) The graphs (a), (b), (c), (d) in Problem 11 of Homework 4.

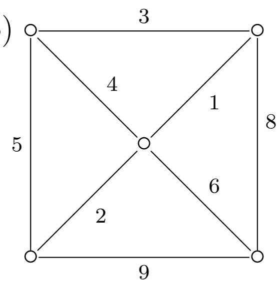
   (d) The graphs (a), (b), (c), (d) in Problem 12 of Homework 4.

4. Solve the MST problem for each weighted graph.

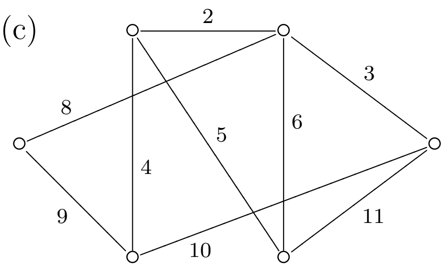
(a)



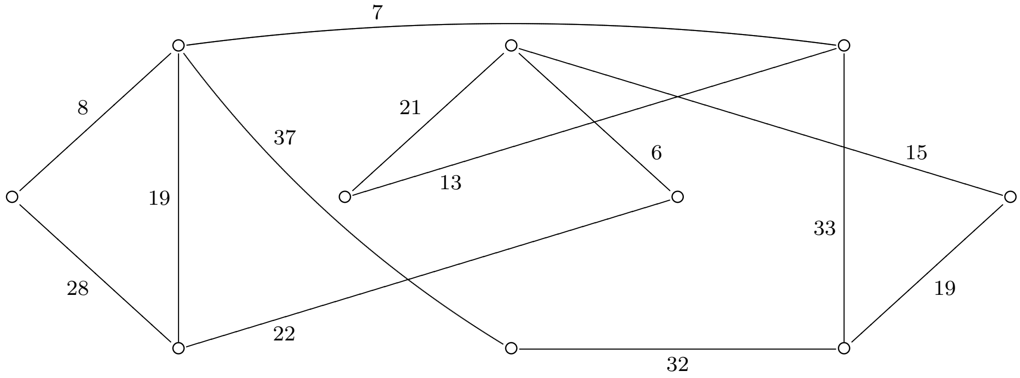
(b)



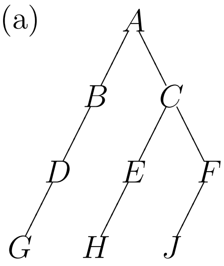
(c)

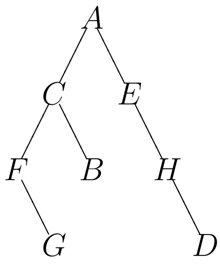


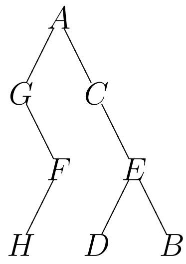
(d)

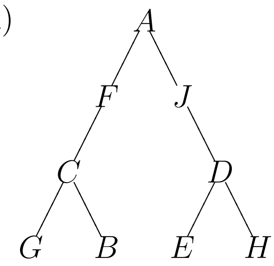


5. Find the output of the pre-order traversal algorithm on each LBT.

(a) 

(b) 

(c) 

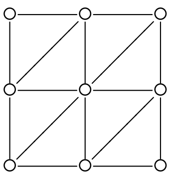
(d) 

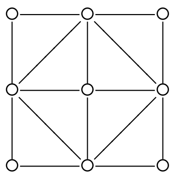
6. Repeat Problem 5 using in-order.

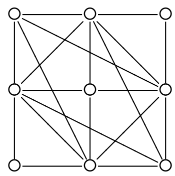
7. Repeat Problem 5 using post-order.

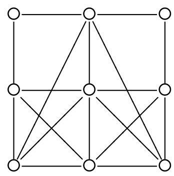
8. Determine an Euler path, Euler circuit, or neither.

   (a) K_5
   (b) K_6
   (c) $K_{2,3}$
   (d) $K_{1,4}$

(e) 

(f) 

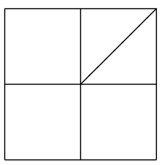
(g) 

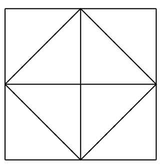
(h) 

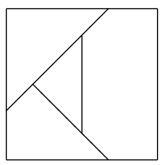
9. Solve the CPP for the same weighted graphs (a), (b), (c), (d) in Problem 4.

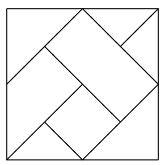
10. Find the chromatic number for the same graphs (a), (b), (c), (d), (e), (f), (g), (h) in Problem 8.

11. For each map, (i) draw the dual graph (ii) find the chromatic number.

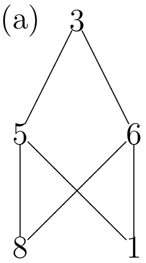
(a) 

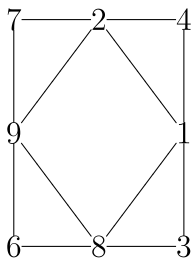
(b) 

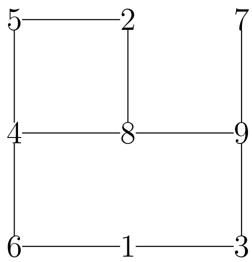
(c) 

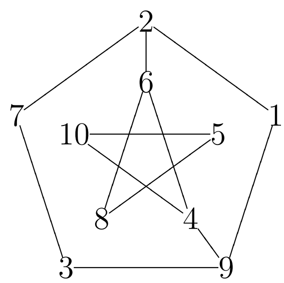
(d) 

12. Draw the spanning tree rooted at vertex 1 using (i) BFS (ii) DFS.

(a) 

(b) 

(c) 

(d) 

Homework 1

1. (a) 17 (b) 42 (c) 110 (d) 359
2. (a) 1011 (b) 1101110 (c) 11111100001 (d) 1000000000000
3. (a) 0111 (b) 0000 (c) 1001 (d) 11101010
4. (a) 10111111 (b) 1011 (c) 1111 (d) 00100000
5. (a) 0001 (b) 1001 (c) 10111111 (d) 11111000
6. (a) T (b) T (c) F (d) F
7. (a) 1100 (b) 1101 (c) 10111010 (d) 00100000
8. (a) 0110 (b) 0101 (c) 11111000 (d) 01010101
9. (a) $\{2, 4, 5\}$ (b) $\emptyset$ (c) $\{5\}$ (d) $\{2, 5, 6, 7, 9\}$
10. (x) A (x) $A \oplus B$ (x) $A \cap B$ (x) $B - A$
11. (a) $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, A\}$ (c) $\{\emptyset, \{x\}, \{\{7\}\}, A\}$ (d) $\{\emptyset\} = \{A\}$
12. (a) 32 (b) 4 (c) 12 (d) 256

Homework 2

1. (a) 3 (b) 21 (c) 14 (d) –6
2. (a) 11 (b) 2 (c) 21 (d) 0
3. (a) 8 (b) 7 (c) 5 (d) 1
4. (a) 9 (b) 2 (c) 26 (d) 3
5. (a) 360 (b) 36 (c) 136 (d) 4941
6. (a) –12, 13 (b) 4, –27 (c) 5, –8 (d) –1433, 418
7. (a) 8 (b) 7 (c) 11 (d) 77
8. (a) T (b) T (c) T (d) F
9. (a) 27 (b) 29 (c) 59 (d) 151
10. (a) 38 (b) 37 (c) 73 (d) 279
11. (a) 50 (b) 12 (c) 263 (d) 19
12. (a) 233 (b) 111 (c) 8 (d) 876

Homework 3

1. (a) 2600 (b) 2600 (c) 8 (d) 16
2. (a) 792 (b) 1287 (c) 211 (d) 1048365

3. (a) 220 (b) 120 (c) 924 (d) 1190

4. (a) 63 (b) 4 (c) 81 (d) 10

5. (a) 0,0,1,1,2,2 (b) –2,–1,1,5,13,29 (c) 0,0,2,6,12,20 (d) 0,1,2,0,1,2

6. (x) $5n+3$ (x) n^2+1 (x) 2^{n+1} (x) $2+n \bmod 2$

7. (a) 32 (b) 1 (c) 65 (d) 20

8. (x) $S(n-1)+S(n-2)$ (x) $2S(n-1)+S(n-2)$ (x) $S(n-1)S(n-2)$
(x) $S(n-1)+S(n-2)+1$

9. (a) 3^n (b) $\frac{9}{7}(4)^n+\frac{5}{7}(-3)^n$ (c) $(2-n)3^n$ (d) $\frac{1}{8}(3)^n-\frac{1}{8}(-5)^n$

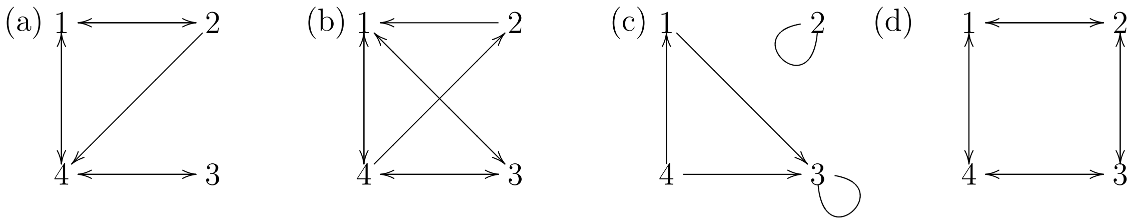
10. (a) $\frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n-\frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$ (b) $\frac{2^n-(-1)^n}{3}$ (c) $(\frac{1+\sqrt{5}}{2})^n+(\frac{1-\sqrt{5}}{2})^n$ (d) $\frac{(1+\sqrt{2})^n-(1-\sqrt{2})^n}{2\sqrt{2}}$

Homework 4

1. (c) $1 \longleftarrow 2 \longleftarrow 3 \longleftarrow 4 \longleftarrow 5$

2. (a) $\begin{bmatrix} 1&1&1&1&1\\ 0&0&0&1&1\\ 0&0&0&1&1\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{bmatrix}$ (b) $\begin{bmatrix} 0&1&1&0\\ 1&0&0&1\\ 1&0&1&0\\ 0&1&0&1 \end{bmatrix}$ (c) $\begin{bmatrix} 0&0&1&0&0\\ 0&0&1&0&1\\ 1&0&0&1&0\\ 0&0&0&0&1\\ 1&1&0&0&0 \end{bmatrix}$ (d) $\begin{bmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix}$

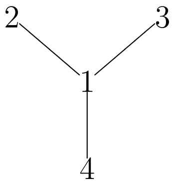
3. (a) $\begin{bmatrix} 1&0&0&0\\ 1&1&0&0\\ 1&0&1&0\\ 1&1&0&1 \end{bmatrix}$ (b) $\begin{bmatrix} 0&1&1&1\\ 0&0&1&1\\ 0&1&0&1\\ 0&0&1&0 \end{bmatrix}$ (c) $\begin{bmatrix} 1&1&1&1\\ 0&1&0&1\\ 0&0&1&0\\ 0&0&0&1 \end{bmatrix}$ (d) $\begin{bmatrix} 0&1&1&1\\ 0&0&0&0\\ 0&1&0&0\\ 0&0&1&0 \end{bmatrix}$

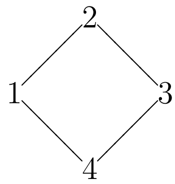
4. 

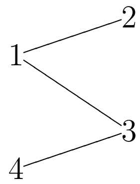
5. (a) $\begin{bmatrix} 1&0&1&1\\ 0&1&0&1\\ 1&1&1&1\\ 0&0&0&1 \end{bmatrix}$ (b) $\begin{bmatrix} 1&0&1&0\\ 0&0&1&0\\ 0&0&1&0\\ 1&0&1&1 \end{bmatrix}$ (c) $\begin{bmatrix} 1&1&1&1\\ 1&0&1&1\\ 1&0&1&1\\ 1&1&1&1 \end{bmatrix}$ (d) $\begin{bmatrix} 1&0&1&1\\ 1&0&1&1\\ 1&0&1&1\\ 1&0&1&0 \end{bmatrix}$

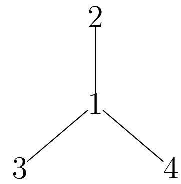
6. (a) $\begin{bmatrix} 0&0&1&1\\ 1&0&1&1\\ 0&0&1&1\\ 0&0&1&1 \end{bmatrix}$ (b) $\begin{bmatrix} 1&0&1&1\\ 1&0&1&1\\ 1&0&1&1\\ 1&0&1&1 \end{bmatrix}$ (c) $\begin{bmatrix} 1&1&1&1\\ 1&1&1&1\\ 1&1&1&1\\ 1&1&1&1 \end{bmatrix}$ (d) $\begin{bmatrix} 1&0&1&0\\ 1&0&1&0\\ 1&0&1&0\\ 1&0&1&0 \end{bmatrix}$

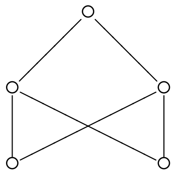
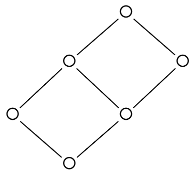
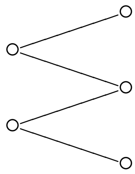
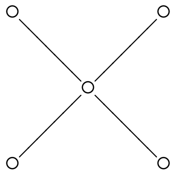
7. (a) $\begin{bmatrix} 1&0&0&0&1\\ 0&1&0&0&1\\ 1&1&1&0&1\\ 1&1&0&1&1\\ 0&0&0&0&1 \end{bmatrix}$ (b) $\begin{bmatrix} 1&1&0&0\\ 0&1&0&0\\ 0&1&1&0\\ 1&1&1&1 \end{bmatrix}$ (c) $\begin{bmatrix} 1&1&0&0\\ 0&1&0&0\\ 1&1&1&0\\ 1&1&1&1 \end{bmatrix}$ (d) $\begin{bmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 1&1&1&0&0\\ 1&1&1&1&0\\ 1&1&1&0&1 \end{bmatrix}$

8. (x) 

(x) 

(x) 

(x) 

9. (d)  (c)  (a)  (b) 

10. (a) $\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

11. (a) $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

12. (a) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

Homework 5

1. (a) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

2. (a) $n(n-1)$ (b) $2mn$ (c) columns times two (d) number of ones

3. (a) d (b) c (c) b (d) b

4. (a) 16 (b) 12 (c) 22 (d) 140

5. (x) AGFHCEDB (x) AFCGBJDEH (x) ACFGBEHD (x) ABDGCEHFJ

6. (x) GHFACDEB (x) GDBAHECJF (x) GCBFAJEDH (x) FGCBAEHD

7. (x) GFBCDHEA (x) GDBHEJFCA (x) GBCFEHDJA (x) HFGDBECA

8. (a) EC (b) NO (c) EP (d) NO (e) EP (f) NO (g) EC (h) EP

9. (a) 51 (b) 49 (c) 71 (d) 320

10. (a) 5 (b) 6 (c) 2 (d) 2 (e) 3 (f) 3 (g) 3 (h) 4

11. (a) 3 (b) 2 (c) 4 (d) 3

–Amin Witno