GRAPH THEORY

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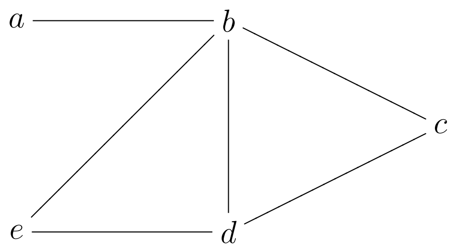
These notes[1] have been prepared for students of Math 352 at Philadelphia University, Jordan. Outline notes are more like a revision. No student is expected to fully benefit from these notes unless they have regularly attended the lectures.

1 Definitions

Definition. A *graph* G is a composite of two finite sets which are commonly labeled $V_G = \{v_1, v_2, \ldots, v_n\}$ and $E_G = \{e_1, e_2, \ldots, e_m\}$. Elements of V_G are called *vertices*, while those of E_G *edges*. An edge e is actually a set of exactly two vertices, e.g., we may write $e = \{v_1, v_2\}$ or simply $e = v_1v_2$.

Example. Let $V_G = \{a, b, c, d, e\}$ and $E_G = \{ab, bc, bd, be, cd, de\}$. We can then represent this graph G by drawing a picture (i.e., a graph) e.g.,



Definition. We call G a *trivial graph* when $E_G = \emptyset$, i.e., when G has only vertices and no edges. Otherwise, when $ab \in E_G$, then we say that the vertices a and b are *adjacent* (to each other). For every vertex $a \in V_G$, we define the *neighbors* of a to be the set of all vertices which are adjacent to a, i.e., $N(a) = \{b \in V_G \mid ab \in E_G\}$. The neighbors of a set of vertices $S \subseteq V_G$ is given by $N(S) = \bigcup_{a \in S} N(a)$.

Example. In the graph pictured above, we have $N(c) = \{b, d\}$ and $N(d) = \{b, c, e\}$, hence $N(\{c, d\}) = \{b, c, d, e\}$.

Definition. By the *degree* of a vertex a, we mean $\deg(a) = |N(a)|$. Then, we define

$$\Delta(G) = \max_{a \in V_G} \deg(a) \quad \text{and} \quad \deg G = \sum_{a \in V_G} \deg(a)$$

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Last Revision: 09–09–2021

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Example. The first example earlier has $\deg(a) = 1$, $\deg(b) = 4$, $\deg(c) = 2$, $\deg(d) = 3$, $\deg(e) = 2$, hence $\deg G = 1 + 4 + 2 + 3 + 2 = 12$. Note here that $\Delta(G) = 4$.

Theorem 1 (Euler's Theorem)**.** The degree of any graph G is twice the number of its edges, i.e., $\deg G = 2\,|E_G|$. In particular, the degree of any graph is an even number.

Proof. Every edge $ab \in E_G$ contributes two to the degree of G, one via $\deg(a)$ and one via $\deg(b)$. Since $\deg G$ is completely determined by the number of edges on each vertex, the result follows. $\triangledown$

Definition. We introduce four families of special graphs.

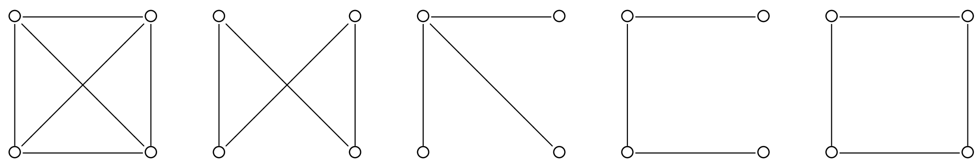
1. A *complete graph* K_n is a graph of n vertices, all of which are adjacent one to another. In particular, K_3 is also called a *triangle*.

2. A *complete bipartite graph* $K_{m,n}$ consists of $m + n$ vertices that are partitioned into two subsets, with m and n elements in each, such that two vertices are adjacent if and only if they do not belong in the same subset.

3. A *path* P_n is a graph with $V_{P_n} = \{v_1, v_2, \ldots, v_n\}$ and $E_{P_n} = \{e_1, e_2, \ldots, e_{n-1}\}$, where each $e_i = v_i v_{i+1}$. In this notation, we say that P_n is a path from v_1 to v_n.

4. A *cycle* C_n ($n \geq 3$) is obtained from the path P_n by adding one more edge: $v_n v_1$. Hence, C_n is also called a *closed path* of n vertices.

Example.



K_4 $\quad$ $K_{2,2}$ $\quad$ $K_{1,3}$ $\quad$ P_4 $\quad$ C_4

Definition. A graph is called *regular* if all its vertices have equal degrees, otherwise *irregular*. In particular, when a graph G has $\deg(a) = d$ for all $a \in V_G$, then G is called *d-regular*.

Example. The graph K_4 is 3-regular, all cycles C_n are regular, and P_5 is irregular.

Definition. If $V_G = \{v_1, v_2, \ldots, v_n\}$, the *degree sequence* of the graph G is the sequence $(\deg(v_i))$ of length n, arranged in non-increasing order. We call a random non-increasing sequence of positive integers *graphical* if we can find a graph matching it as its degree sequence.

Example. The degree sequence of P_5 is $(2, 2, 2, 1, 1)$, hence the sequence $(2, 2, 2, 1, 1)$ is graphical. On the other hand, we will see that the sequence $(4, 3, 2, 2, 2)$ is not graphical.

Algorithm 2 (Graphical Degree Sequence)**.** Given a non-increasing sequence of positive integers, we determine graphical or not graphical.

1. Delete the first integer, say k.

2. From what remains, subtract the first k numbers each by one. If we get negative numbers, the sequence is not graphical. If we get all zeros, the sequence is graphical.

3. Rearrange the resulting sequence in non-increasing order, if necessary, and repeat the above steps until a conclusion is obtained.

Example. We illustrate the algorithm on the sequence $(3, 2, 2, 1, 1, 1)$:

$$3, 2, 2, 1, 1, 1 \to 1, 1, 0, 1, 1 \to 1, 1, 1, 1, 0 \to 0, 1, 1, 0 \to 1, 1, 0, 0 \to 0, 0, 0$$

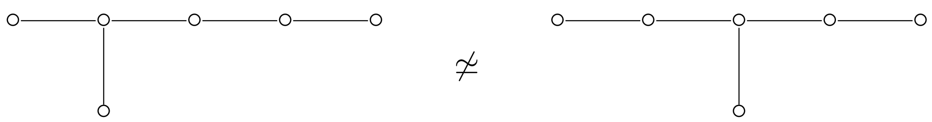
We finish with all zeros, so the sequence is graphical.

Definition. Two graphs G and H are *isomorphic* to each other, written $G \simeq H$, if there exists a bijection $f : V_G \to V_H$ such that $ab \in E_G$ if and only if $f(a)f(b) \in E_H$ for all $a, b \in V_G$.

Example. Note that $K_3 \simeq C_3$, so both are called triangles, and that $K_2 \simeq K_{1,1} \simeq P_2$.

It follows that $G \simeq H$ if and only if both graphs can be represented by identical pictures. In such case, it is necessary for G and H to have the same number of vertices, the same number of edges, and the same degree sequences. However, none of these conditions is sufficient for isomorphism.

Example. Here are two graphs with identical degree sequence $(3, 2, 2, 1, 1, 1)$:



To prove that they are not isomorphic to each other, note that each graph has a unique vertex of degree 3, call them a and w. So any bijection f must have $f(a) = w$ to be an isomorphism. This is impossible since $N(a)$ consists of vertices with degree sequence $(2, 1, 1)$, whereas for $N(w)$ we have $(2, 2, 1)$.

Definition. A graph H is a *subgraph* of the graph G if $V_H \subseteq V_G$ and $E_H \subseteq E_G$. We denote this relation by $H \subseteq G$ and may say that G *contains* H. By abuse of notation, we also write $H \subseteq G$ when we really mean that G contains a subgraph which is isomorphic to H.

Example. We have $P_3 \subseteq C_3$ and $K_{2,2} \subseteq K_{2,4}$.

Definition. A graph G is *connected* if there is a path from any vertex to any other vertex in G, otherwise *disconnected*. A *component* of G is a maximal connected subgraph of G. Hence, a graph is disconnected if and only it has more than one component.

Definition. An edge of G is called a *bridge* if removing it would increase the number of components of G. So if G is connected, removing a bridge would make G disconnected.

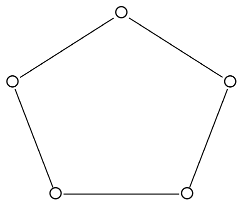
Example. Note that in P_4 every edge is a bridge, whereas C_4 has no bridge.

Theorem 3. If G is a connected graph, then $|E_G| \geq |V_G| - 1$.

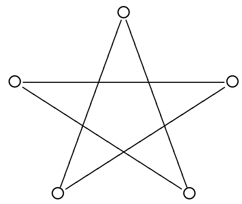
Proof. Assume that G is connected with $|V_G| = n \geq 2$, so we can have at least one edge v_1v_2. Since G is connected, if $n \geq 3$, then there is another vertex $v_3 \in N(\{v_1, v_2\})$, i.e., with either v_1v_3 or $v_2v_3 \in E_G$. Again, if $n \geq 4$, there exists $v_4 \in N(\{v_1, v_2, v_3\})$ with v_1v_4, v_2v_4, or $v_3v_4 \in E_G$. By the time we reach v_n, we will have encountered $n - 1$ edges of G. $\triangledown$

Definition. The *complement* of a graph G is the graph $\overline{G}$, where $V_{\overline{G}} = V_G$ and $ab \in E_{\overline{G}}$ if and only if $ab \notin E_G$.

Example. We show the picture of C_5 next to its complement:



$C_5 \quad \to \quad \overline{C}_5$



Theorem 4. If G is disconnected, then $\overline{G}$ is connected.

Proof. Suppose that G is disconnected and consider two vertices a and b. Remember that $V_{\overline{G}} = V_G$. If a and b are not adjacent in G, then they are in $\overline{G}$. If $ab \in E_G$, choose a vertex c belonging to a component of G not containing a, b. Such a vertex c exists since G is disconnected. Then $ac \notin E_G$ and $cb \notin E_G$, and they form a path from a to b in $\overline{G}$. As a and b are arbitrary, this proves that $\overline{G}$ is connected. $\triangledown$

Definition. A graph G is *self-complementary* when $\overline{G} \simeq G$. The preceding theorem requires a self-complementary graph to be connected.

Example. Observe from the previous example that $C_5 \simeq \overline{C}_5$. Another example is P_4.

Recall from Linear Algebra that if M denotes an arbitrary matrix, we write $[M]_{ij}$ to refer to the (i, j) entry in M, i.e., the entry in the ith row and jth column of M.

Definition. Suppose that $V_G = \{v_1, v_2, \ldots, v_n\}$. The *adjacency matrix* of the graph G is the $n \times n$ matrix A given by

$$[A]_{ij} = \begin{cases} 1 & \text{if} \quad v_i v_j \in E_G \\ 0 & \text{if} \quad v_i v_j \notin E_G \end{cases}$$

Example. The adjacency matrix of C_4, where $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$, is given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

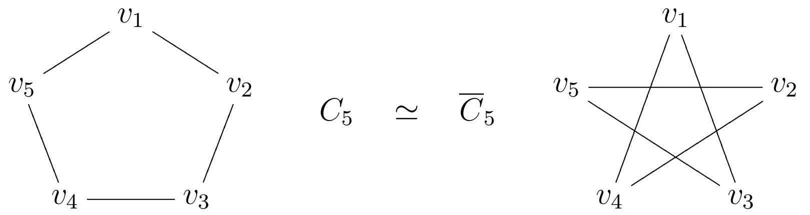
Definition. A *permutation matrix* is a square matrix obtained from the identity matrix by reordering its rows. A known fact is that every permutation matrix P belongs to the family of orthogonal matrices, i.e., that $P^{-1} = P^T$.

Example. Let P be the permutation matrix obtained from the identity matrix by permuting its rows in the order of $(1, 3, 5, 2, 4)$, i.e.,

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Theorem 5. Let the graphs G and H have adjacency matrices A and B, respectively. Then $G \simeq H$ if and only if $B = PAP^T$ for some permutation matrix P.

Example. We look again at the fact that $C_5 \simeq \overline{C}_5$, this time with the vertices labeled:



An obvious bijection is one which permutes the vertex indices $(1, 2, 3, 4, 5)$ to $(1, 3, 5, 2, 4)$, thus the permutation matrix P given earlier. Denote the corresponding adjacency matrices A and B, respectively, and we verify that $PAP^T = B$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Definition. Suppose that $V_G = \{v_1, v_2, \ldots, v_n\}$ and $E_G = \{e_1, e_2, \ldots, e_m\}$. Then the *incidence matrix* of the graph G is the $n \times m$ matrix Z given by

$$[Z]_{ij} = \begin{cases} 1 & \text{if} \quad v_i \in e_j \\ 0 & \text{if} \quad v_i \notin e_j \end{cases}$$

Example. The incidence matrix of P_4, where $E = \{v_1v_2, v_2v_3, v_3v_4\}$, is given by

$$Z = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Definition. Let G be a graph with vertices $v_1, v_2, \ldots, v_n$. The *degree matrix* of G is the $n \times n$ diagonal matrix D given by $[D]_{ii} = \deg(v_i)$.

Theorem 6. Suppose that the adjacency matrix A and incidence matrix Z have been given for the same graph G. Then $ZZ^T = A + D$, where D is the degree matrix of G.

Proof. For $i \neq j$, we have

$$[ZZ^T]_{ij} = \sum_{k \geq 1} [Z]_{ik}[Z^T]_{kj} = \sum_{k \geq 1} [Z]_{ik}[Z]_{jk}$$

If $v_iv_j \in E$, then there is exactly one value of k for which $[Z]_{ik} = [Z]_{jk} = 1$, while the rest either $[Z]_{ik} = 0$ or $[Z]_{jk} = 0$. In that case, both ZZ^T and $A + D$ have 1 in their (i, j) entries. If $v_iv_j \notin E$, no such k exists, and that entry will be 0 in both.

For $i = j$, we have

$$[ZZ^T]_{ii} = \sum_{k \geq 1} [Z]_{ik}[Z^T]_{ki} = \sum_{k \geq 1} [Z]_{ik}^2 = \sum_{k \geq 1} [Z]_{ik}$$

(Note the exponent 2 is redundant when squaring 0 or 1.) So $[ZZ^T]_{ii}$ counts the number of vertices adjacent to v_i, which agrees with the diagonal entry $[A + D]_{ii} = [D]_{ii}$. $\triangledown$

2 Trees

Definition. A graph G is *cyclic* if $C_n \subseteq G$ for some $n \geq 3$. A graph which contains no cycles is called *acyclic*. A *tree* is a connected acyclic graph.

Example. K_4 is a cyclic graph, and P_4 is acyclic. Since also connected, P_4 is a tree.

Theorem 7. Let G be a connected graph. The following are equivalent conditions for G to be a tree:

1. G is acyclic.

2. Every edge in G is a bridge.

3. The size of G is determined by $|E_G| = |V_G| - 1$.

4. There is a unique path between any two vertices in G.

Hence, a connected graph is a tree if and only if any one of the above conditions holds.

Proof. We will prove first the chain of implications $\neg(4) \to \neg(1) \to \neg(2) \to \neg(3)$: Being connected, we always have a path from a to b, arbitrary vertices. If such a path is not unique, then two distinct paths from a to b make a closed path, hence G is cyclic. Moreover, every edge belonging to a cycle is a non-bridge. And if a non-bridge edge is removed, the resulting subgraph H remains connected. By Theorem 3, we have $|E_H| \geq |V_H| - 1 = |V_G| - 1$, hence $|E_G| = 1 + |E_H| > |V_G| - 1$.

We are now left with proving $\neg(3) \to \neg(4)$: In the proof of Theorem 3 we show the existence of a connected subgraph T with $V_T = V_G$ and $|E_T| = |V_G| - 1$. If $\neg(3)$ holds, then $|E_G| > |V_G| - 1$, i.e., there exists an edge $ab \in E_G$ but $ab \notin E_T$. However, T connected means that we already have a path from a to b in T. Thus the edge ab is a second such path, yielding $\neg(4)$. $\triangledown$

Definition. In a graph, a vertex of degree one is called a *leaf.*

Theorem 8. Every tree has a leaf. Furthermore, if n_i denotes the number of vertices of degree i, then the number of leaves is given by

$$2 + \sum_{i \geq 3} (i-2) n_i$$

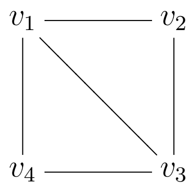
Proof. Let $(d_1, d_2, \ldots, d_n)$ be the degree sequence of a tree, all positive since connected. With $n-1$ edges, Euler's theorem gives $\sum d_i = 2n - 2 < 2n$, hence a leaf exists. The number of leaves is minimum in the case $(2, 2, \ldots, 2, 1, 1)$. The existence of a vertex of degree $i \geq 3$ will reduce the number of twos in the sequence and add the number of leaves—exactly $i - 2$ of them. This yields the claimed formula. $\triangledown$

Definition. A *spanning tree* of a graph G is a tree $T \subseteq G$ with $V_T = V_G$.

Example. P_4 is a spanning tree of C_4 as well as of K_4. Another spanning tree for K_4 is $K_{1,3}$.

Theorem 9 (The Matrix Tree Theorem)**.** Let G be a connected graph with labeled vertices, adjacency matrix A, and degree matrix D. Then any cofactor of the matrix $D - A$ will give the number of spanning trees of G.

Example. The graph G is given, together with the associated matrix $D - A$.



$$G \quad \to \quad D - A = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

Recall that the cofactor $C_{i,j}$ of a square matrix M is $(-1)^{i+j}$ times the determinant of the matrix obtained from M by removing the ith row and jth column. For example, using $C_{3,1}$ we conclude that G has 8 spanning trees:

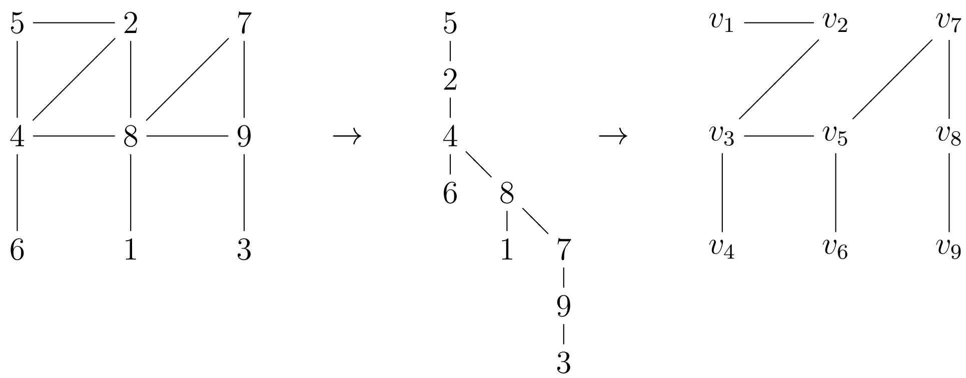
$$C_{3,1} = (+) \det \begin{bmatrix} -1 & -1 & -1 \\ 2 & -1 & 0 \\ 0 & -1 & 2 \end{bmatrix} = 8$$

Definition. Any vertex of a tree can be designated as the *root* with respect to which every other vertex is pictorially represented below the root from which, by uniqueness of path, each level down reflects the length of the path to each vertex.

Algorithm 10 (Depth-First Search)**.** Given a connected graph G with pre-ordered vertices $v_1, v_2, \ldots, v_n$, we produce a rooted spanning tree.

1. Choose a vertex v_k as the root.
2. Directly below this root, place the one adjacent vertex of least index which has never been selected, called this the subroot.
3. Repeat Step 2 with respect to the new subroot. If there is no adjacent vertices left, backtrack upward to the most immediate parent and select another subroot from it.
4. Repeat until all vertices have been traversed.

Example. We display the output upon applying DFS algorithm using the graph given on the left, by choosing the vertex 5 as the root. We also show how the unrooted spanning tree looks like in the original formation.



Algorithm 11 (Breadth-First Search)**.** Given a connected graph G with pre-ordered vertices $v_1, v_2, \ldots, v_n$, we produce a rooted spanning tree.

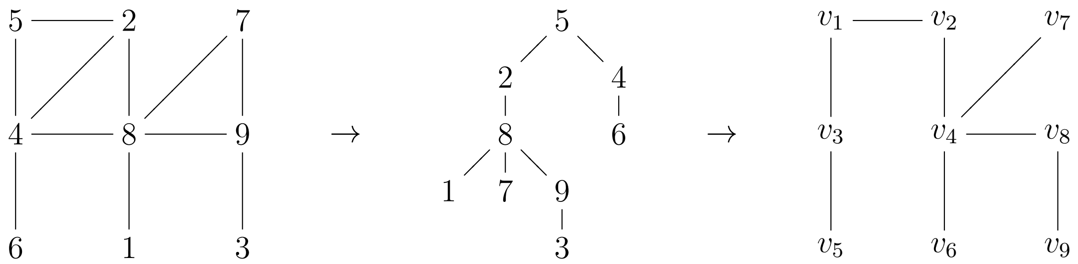
1. Choose a vertex v_k as the root.

2. Directly below this root, place all the adjacent vertices that have never been selected, ordered from left to right according to their indices. These are the immediate subroots.

3. Repeat Step 2 with respect to the new subroots, one at a time from left to right.

4. Repeat until all vertices have been traversed.

Example. We display the output upon applying BFS algorithm using the graph given on the left, by choosing the vertex 5 as the root. We also show how the unrooted spanning tree looks like in the original formation.



Definition. A graph is *weighted* if every edge is associated with a positive numerical value, called the *weight*. A *minimal spanning tree* of a weighted graph is a spanning tree with the least total weight. While a minimal spanning tree may not be unique, the least value of its total weight, by definition, is unique.

Algorithm 12 (Kruskal's Minimal Spanning Tree)**.** Given a connected weighted graph G, we generate a minimal spanning tree T.

1. Start with an edge e_1 of least weight to start with $E_T = \{e_1\}$.

2. Choose a new edge from E_G of least weight to be added to T, provided that the resulting T remains acyclic.

3. Repeat until $|E_T| = |V_G| - 1$.

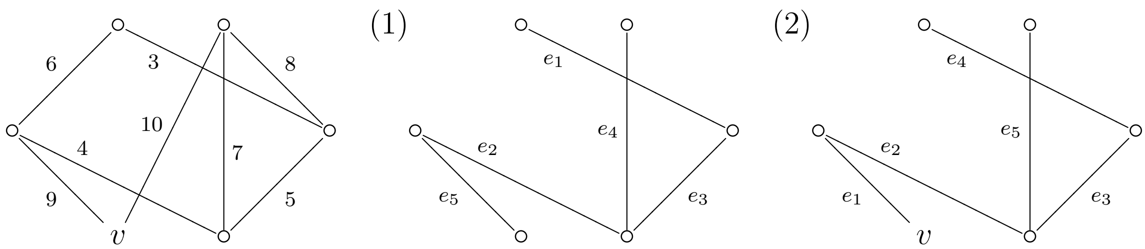
Algorithm 13 (Prim's Minimal Spanning Tree)**.** Given a connected weighted graph G, we generate a minimal spanning tree T, starting at a specified vertex v.

1. Any vertex v can be selected to start with $V_T = \{v\}$.

2. Choose a new edge from E_G of least weight to be added to T, provided that the resulting T remains acyclic and connected.

3. Repeat until $|E_T| = |V_G| - 1$.

Example. For comparison, we display the output of (1) Kruskal's and (2) Prim's minimal spanning tree, respectively, out of the same graph given on the left:



The MST sequence is $(3, 4, 5, 7, 9)$ for Kruskal's, and $(9, 4, 5, 3, 7)$ for Prim's, either one of which yields the total weight of 28.

3 Walking

Definition. A *walk* is a sequence of continuous edges of the form $v_1v_2, v_2v_3, \ldots, v_nv_{n+1}$. In this notation, we have a walk of *length* n from v_1 to v_{n+1}. Unless the walk is a path, these edges are not assumed distinct, and neither are the vertices. The walk is a *closed walk* if $v_{n+1} = v_1$. We loosely treat a walk as a graph containing these edges and the corresponding vertices.

Example. A path P_n can be considered as a walk of length $n-1$, while C_n a closed walk of length n.

Theorem 14. If G is a closed walk of odd length, it contains a cycle of odd length.

Proof. A closed walk of length three is none other than C_3, so the claim is true. We proceed by induction, assuming the theorem has been proved if the length is less than n. If G is the walk $v_1v_2, v_2v_3, \ldots, v_nv_1$ with no repeated vertex, then $G \simeq C_n$ and we are done. Suppose now $v_i = v_{i+j}$. Then G is really the union of two closed walks: the one from v_i to v_{i+j}, and the walk from v_1 to v_i joined by that from v_{i+j} to v_1. One of the two must have an odd length because their sum is odd, and which is clearly less than n. By our hypothesis, that walk contains a cycle of odd length. $\triangledown$

Theorem 15. Let A be the adjacency matrix of a graph G with vertices $v_1, v_2, \ldots, v_n$. The number of walks of length k from v_i to v_j is then given by $[A^k]_{ij}$.

Proof. Clearly $[A]_{ij} = 1$ if and only if $v_iv_j \in E_G$, in which case the edge v_iv_j is the only walk of length one from v_i to v_j. We proceed by induction. Since a walk of length $k+1$ from v_i to v_j consists of a walk of length k from v_i to an intermediate vertex v_m, which must be adjacent to v_j, then we may express the total number of such walks by

$$\sum_{m=1}^{n} [A^k]_{im}\,[A]_{mj} = [A^{k+1}]_{ij}$$

and the induction is complete. $\triangledown$

Theorem 16. Let A denote the adjacency matrix of a graph G. Then the number of unlabeled triangles contained in G is given by one-sixth of the trace of A^3, i.e.,

$$\frac{1}{6}\sum_{i\geq 1} [A^3]_{ii}$$

Proof. A closed walk of length 3 is a triangle. Since the vertices of a labeled triangle can be permuted in 6 different ways, the result follows after the preceding theorem. $\triangledown$

Definition. The *distance* between two vertices a and b, denoted by $d(a, b)$, is the length of the shortest walk from a to b, if it exists; otherwise let $d(a, b) = \infty$.

Example. If a and b are adjacent, then $d(a, b) = 1$. By definition, we see that $d(a, a) = 0$ for any vertex a. Note that the shortest walk is necessarily a path.

Definition. Suppose that $V_G = \{v_1, v_2, \ldots, v_n\}$. The *distance matrix* of the graph G is the $n \times n$ matrix D given by $[D]_{ij} = d(v_i, v_j)$.

Example. The distance matrix of P_4, when labeled in the standard way, is given by

$$D = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}$$

Algorithm 17 (From A to D)**.** With computer's help, we can retrieve the distance matrix D from a given adjacency matrix A.

1. Compute the matrices $A, A^2, A^3, \ldots, A^n$, where $n \times n$ is the size of A.
2. Set $[D]_{ii} = 0$, and for $i \neq j$ let $[D]_{ij} = k$, the least exponent for which $[A^k]_{ij} \neq 0$. If no such k exists, set $[D]_{ij} = \infty$.

Definition. The *diameter* of a graph G, denoted by $d(G)$, is the largest possible distance between two vertices in G. Thus $d(G)$ is the largest entry in the distance matrix of G.

Example. We have $d(K_4) = 1$ and $d(P_4) = 3$. In general $d(G) = 1$ if and only if G is complete. Note also that $d(G) = \infty$ if and only if G is disconnected.

Theorem 18. If $d(G) \geq 3$, then $d(\overline{G}) \leq 3$.

Proof. Assume $d(G) \geq 3$ and let $v, w \in V_G = V_{\overline{G}}$. We will show that $d(v, w) \leq 3$ in $\overline{G}$.

We know there are a and b for which $d(a, b) \geq 3$ in G. It is then false if both $av, vb \in E_G$ or both $aw, wb \in E_G$. So we have four cases to consider:

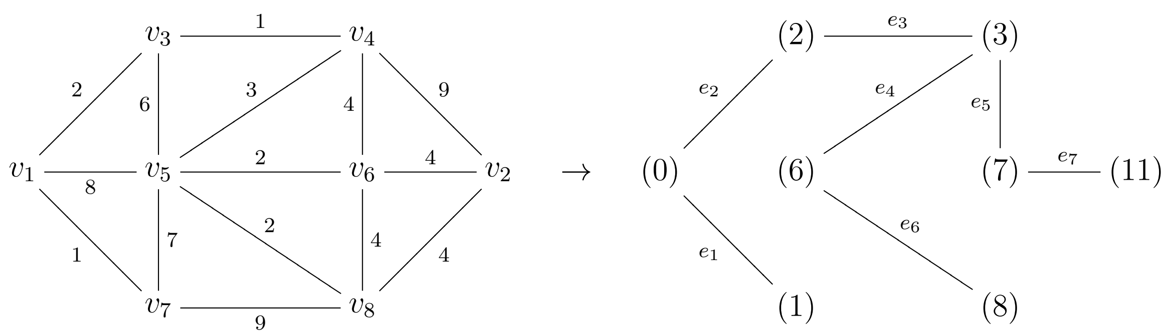
(1) $E_{\overline{G}}$ contains av and aw, hence $d(v, w) \leq 2$ in $\overline{G}$.
(2) $E_{\overline{G}}$ contains vb and wb, again $d(v, w) \leq 2$ in $\overline{G}$.
(3) $E_{\overline{G}}$ contains av and wb. Since we also have $ab \in E_{\overline{G}}$, then $d(v, w) \leq 3$ in $\overline{G}$.
(4) $E_{\overline{G}}$ contains vb and aw. Again with $ab \in E_{\overline{G}}$, we have $d(v, w) \leq 3$ in $\overline{G}$. ▽

Definition. If G is a weighted graph, we redefine the distance $d(a, b)$ to be the least total weight of all possible walks from a to b.

Algorithm 19 (Dijkstra)**.** For a fixed vertex a in a weighted graph G, we determine $d(a, x)$ for all $x \in V_G$, i.e., we get the entire Row (a) of the distance matrix of G.

1. Denote by S the set of vertices s which have already been labeled by (v_s, W_s), where v_s is a vertex of G and W_s is an integer. Initially, we set $S = \{a\}$ and label a by (0).
2. For each vertex $x \in N(S)$, say adjacent to $y \in S$ whose edge xy has weight W, calculate the number $W_x = W + W_y$. Add to S the vertex x for which the corresponding W_x is least possible, and label x by (W_x). Hence, $W_x = d(a, x)$.
3. Repeat until all vertices have been labeled.

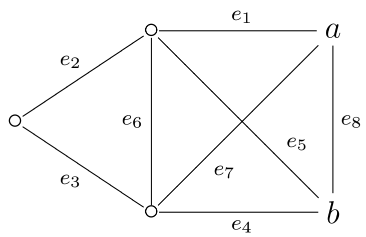
Example. Using Dijkstra's algorithm, we compute $d(v_1, x)$ for every $x \in V_G$ as follow.



In the end, we get Row (1) of the distance matrix of G, i.e., $[0, 11, 2, 3, 6, 7, 1, 8]$, e.g., $d(v_1, v_2) = 11$ and $d(v_1, v_8) = 8$. The resulting spanning tree also shows the unique shortest path from v_1 to any other vertex x. Note that obtaining this spanning tree is similar to Prim's algorithm, but instead of searching for a least weight in each iteration, we consider the least accumulative weight relative to the starting vertex.

Definition. An *Euler walk* in a connected graph G is a walk through all the edges of G without repeating any of them. If an Euler walk is closed, we shall call it an *Euler circuit.*

Example. The following graph has an Euler walk from a to b following the labeled edges $e_1, e_2, \ldots, e_8$, in this order.



Theorem 20. A connected graph has an Euler walk from a to b, with $a \neq b$, if and only if a and b are the only vertices whose degrees are odd. The graph has an Euler circuit if and only if all vertices have even degrees.

Proof. Consider a vertex v with $\deg(v) = d$. At some point during the walk, we will run into v and out via another edge. If $d > 2$, this process will repeat, for as long as there are untrodden edges containing v. This shows the necessity that d be even, unless v is our starting point, or last destination, in which case d may be odd.

To prove sufficiency, assume first that every vertex in a graph G has an even degree. Consider a path P of maximum length from v to w. Since $\deg(w)$ is at least two, w is adjacent to another vertex already contained in P, else we could extend P to a longer path. Hence G contains a cycle C. Since every vertex in C has an even degree, so does the subgraph whose edges are in $G - C$. Repeating the argument, we see that the edges in G can be thought of as the union of cycles none of which is disjoint from the rest.

We finish off by induction. One cycle is itself an Euler circuit. Assume that the union of n such cycles has an Euler circuit, call it E. With one more cycle C, which meets E at a vertex x, we have an Euler circuit for $E \cup C$ by starting at x, circuiting around E back to x, and cycling around C back to x.

Lastly, if $\deg(a)$ and $\deg(b)$ are the only odd degrees in G, we add one more edge, i.e., ab into G (perhaps making G a multigraph) so that every vertex now has an even degree. We have shown that an Euler circuit exists for this extended graph. Hence, without this extra edge, we could Euler walk from a and terminate at b. $\triangledown$

Definition. The *Chinese Postman Problem* asks for the minimal closed walk going through every edge in a weighted graph. If exists, an Euler circuit would certainly be the desired solution, otherwise such a walk would have to repeat one or more edges.

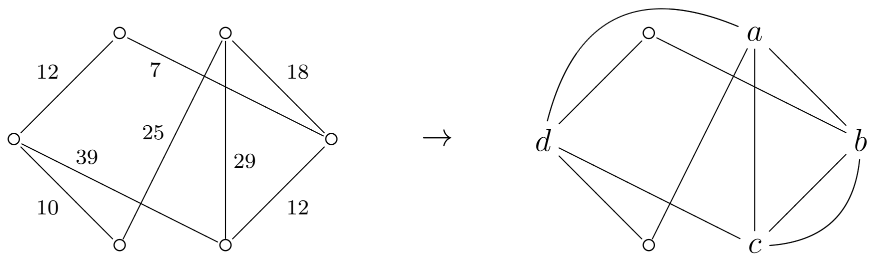
Algorithm 21 (Chinese Postman Problem)**.** We determine the minimal closed walk containing every edge in a given weighted graph.

1. Identify all vertices which have odd degrees. By Euler's theorem, their number is even.

2. Pair up these odd vertices two by two, say $\{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_n, b_n\}$, in such a choice that minimizes the sum $\sum d(a_i, b_i)$.

3. Note that if we add "imaginary" edges $a_i b_i$, the resulting graph would have all vertices of even degree. Hence, solution to the Chinese postman problem is the Euler circuit on this imaginary graph, which is really a walk through all the edges of G plus the repetition paths from a_i to b_i through the existing edges.

Example. The graph on the left has four vertices of odd degree which we label a, b, c, d, shown on the right together with the imaginary edges $\{a, d\}$ and $\{b, c\}$:



The pairing $\{a, d\}$ and $\{b, c\}$ is selected out of three possible choices:

$$
\begin{aligned}
d(a,b) + d(c,d) &= (18) + (12 + 7 + 12) = 49 \\
d(a,c) + d(b,d) &= (29) + (7 + 12) = 48 \\
d(a,d) + d(b,c) &= (25 + 10) + (12) = 47
\end{aligned}
$$

The minimal walk solution goes through all these edges plus the repetition paths from a to d and from b to c, of total weight $152 + 47 = 199$.

Definition. A *Hamilton cycle* in a graph G is a cycle $C_n \subseteq G$ such that $n = |V_G|$, i.e., a closed walk through all the vertices in G without repeating any of them except, of course, the starting vertex. If a Hamilton cycle exists, then G is called a *Hamilton graph.*

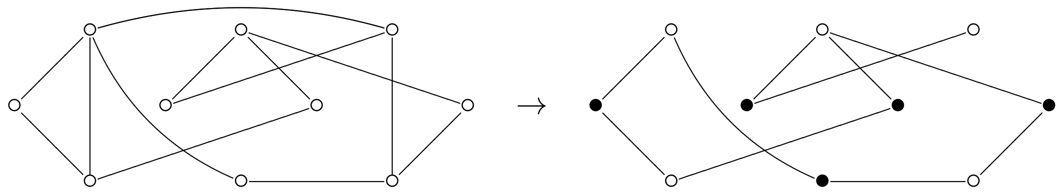
Example. K_4 is a Hamilton graph, while P_4 is not. Note that a Hamilton graph is necessarily connected.

Theorem 22. If G is a Hamilton graph, then G contains no leaves, no bridges, and no cut vertices. A *cut vertex* is one that disconnects the graph when removed.

Proof. A Hamilton cycle (or any cycle) is 2-regular, hence every vertex in G needs to have degree at least two. Now a bridge is the only path between two components, so a closed walk through both components must cross the bridge twice, hence not a cycle. Similarly, a closed walk through a cut vertex must repeat the vertex. $\triangledown$

Example. We can see that if a vertex has degree two, then any Hamilton cycle must contain both its edges. Using this fact, we prove that the given graph on the left is not

Hamilton: There are five vertices of degree two, indicated on the right by black dots. So all the edges where these dots are on must be part of the Hamilton cycle, if exists:



However, the incomplete solution has a vertex of degree 3, so no Hamilton cycle (or any cycle) can have this subgraph.

Theorem 23. Let G be a connected graph with n vertices. If $\deg(a) \geq \frac{n}{2}$ for all $a \in V_G$, then G is a Hamilton graph.

Proof. Let $|V| = n$ and P be a path of maximum length contained in G, given by $v_1v_2, v_2v_3, \ldots, v_{k-1}v_k$. Having maximum length means that the neither v_1 nor v_k is adjacent to any other vertex outside P. Since $\deg(v_1) \geq \frac{n}{2}$, there are at least this many vertices in P adjacent to v_1, and similarly to v_k. By the pigeonhole principle, we can find v_j, with $2 \leq j \leq k$, such that both $v_1v_j, v_{j-1}v_k \in E$. This gives us a cycle given by the closed walk

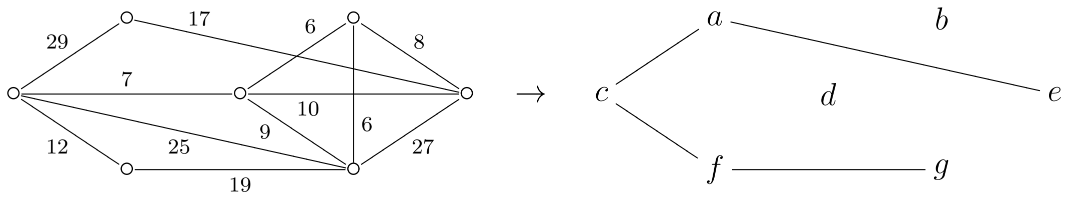
$$v_1v_2, \ldots, v_{j-2}v_{j-1}, v_{j-1}v_k, v_kv_{k-1}, \ldots, v_{j+1}v_j, v_jv_1$$

Any additional vertex connected to this cycle would contradict the maximality of the length of P. Hence $k = n$, i.e., we have a Hamilton cycle. $\triangledown$

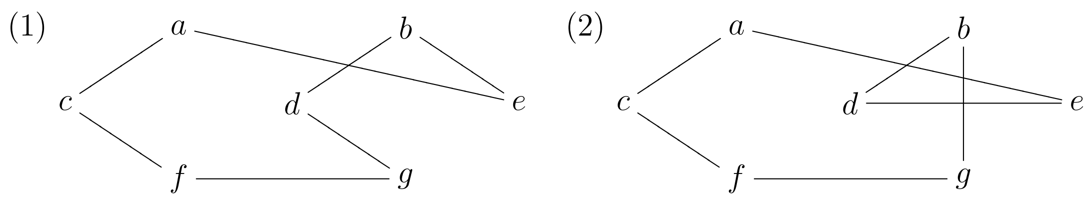
Definition. The *Traveling Salesman Problem* asks for a Hamilton cycle of least total weight in a given weighted graph. One way to solve the problem is to try out all possible Hamilton cycles, which is reasonable only in small cases.

Algorithm 24 (Traveling Salesman Problem)**.** We find all Hamilton cycles of a given weighted graph in order to choose one with least total weight. We assume that the graph is relatively small or has mostly vertices of degree two.

Example. The given graph on the left has two vertices of degree two, labeled a and f as shown on the right with a sketch of the incomplete Hamilton cycle:



If we start from e and complete the cycle, there remain b and d. This gives 2! different Hamilton cycles, i.e., (1) $eacfg - db - e$ and (2) $eacfg - bd - e$.



We check that the second Hamilton cycle has the lesser total weight of 99.

4 Coloring

Definition. The set notation $X \sqcup Y$ is the *disjoint union* of X and Y, i.e., the ordinary set union $X \cup Y$ but with the assumption that $X \cap Y = \emptyset$. A graph G is *bipartite* if $V_G = X \sqcup Y$ such that two vertices are adjacent only if exactly one of them belongs to X (and the other to Y). In this case, we call X and Y the *bipartition subsets* of G.

It is clear that a bipartite graph is simply a subgraph of $K_{m,n}$.

Example. The cycle C_4 with edges $\{ab, bc, cd, da\}$ is bipartite with $X = \{a, c\}$ and $Y = \{b, d\}$. Note that if G is disconnected, then G is bipartite if and only if each component is bipartite.

Theorem 25. A graph G is bipartite if and only if there is no cycle of odd length.

Proof. If G is bipartite and $v_1v_2, v_2v_3, \ldots, v_nv_1$ is a cycle, then by definition we deduce that $v_1, v_3, v_5, \ldots$ all belong to the same bipartition subset. In particular, $v_1v_n \in E_G$, so n must be even. Conversely, let G contain no odd cycle. We assume that G is connected, or else it suffices to consider a component of G. Fix a vertex v and set

$$X = \{w \in V_G \mid d(v, w) \text{ is even}\} \quad \text{and} \quad Y = \{w \in V_G \mid d(v, w) \text{ is odd}\}$$

It is clear that $V_G = X \sqcup Y$. Now if $a, b \in X$, there is a path of even length from v to each of a and b. So if $ab \in E_G$, then the union of ab and these two paths makes a closed walk of odd length, and G would have an odd cycle according to Theorem 14. This contradiction proves that if $a, b \in X$, then $ab \notin E_G$. It is also not hard to show by similar argument that $ab \notin E_G$ if $a, b \in Y$, thus G is indeed bipartite. $\triangledown$

Definition. Let G be a bipartite graph with $V_G = X \sqcup Y$, and assume that $|X| \le |Y|$. Every edge $ab \in E_G$ can be viewed as a relation $(a, b) \in X \times Y$. We say that G has a *complete matching* when we can find a set of these relations (i.e., edges) that form a one-to-one function from X to Y.

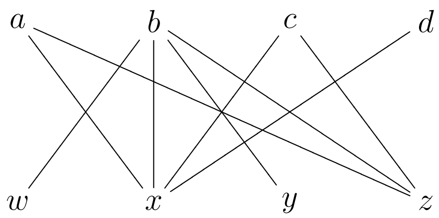
In the case $|X| = |Y|$, a complete matching is also called a *perfect matching.*

Theorem 26 (Hall's Theorem)**.** Let G be bipartite with $V_G = X \sqcup Y$. Then G has a complete matching if and only if $|S| \le |N(S)|$ for every subset $S \subseteq X$.

Proof. The necessary condition $|S| \le |N(S)|$ is required by the definition of one-to-one function. Now assume that this condition is satisfied and assume also, by induction, that we have a matching M contaning all of X except one vertex $v \in X$. Observe that if we have path $P = \{v_1v_2, v_2v_3, \ldots, v_{2n-1}v_{2n}\}$ such that $v_kv_{k+1} \in M$ if and only if k is even, and that both $v_1, v_{2n} \notin M$, then replacing the edges in $M \cap P$ by those in $P - M$ will produce a new matching M' which contains one more vertex from each X and Y. To complete the induction, we will produce such an "alternating" path from $v_1 = v$.

Since $N(v)$ is non-empty, we can find $w_1 \in Y$ which is adjacent to v. If $w_1 \notin M$ then vw_1 is such path, we are done. Else, there is an edge $v_1w_1 \in M$. Since $|N(\{v, v_1\})| \ge 2$, we have w_2, adjacent to either v or v_1. If $w_2 \notin M$, again we have an alternating path from v to w_2, so assume there is another edge $v_2w_2 \in M$. Continuing in this way, seeing that $|N(\{v, v_1, v_2, \ldots v_k\})| > k$ in each step, we will exhaust the vertices in $Y \cap M$, forcing a vertex $w \notin M$ to which there is an alternating path from v. $\triangledown$

Example. In the bipartite graph below, a complete matching is not possible because we have a counter-example with $S = \{a, c, d\}$, for which $|S| > |N(S)|$.



Theorem 27. Suppose that a regular graph G is bipartite with $V_G = X \sqcup Y$. Then $|X| = |Y|$ and G has a perfect matching.

Proof. Assume that G is d-regular with $|X| = n$, so that $|E_G| = dn$. Now if $|Y| = m$, then we have $|E_G| = dm$, hence it is necessary that $m = n$. Furthermore, any set S of k vertices in X corresponds to dk edges which are connected to k vertices in Y. We then have $|N(S)| = |S|$ and a perfect matching by Hall's theorem. $\triangledown$

Definition. In graph theory, the term *coloring* means assigning a color to every vertex in G in such a way that adjacent vertices have distinct colors. The *chromatic number* $\chi(G)$ stands for the least possible number of colors needed to perform this task.

Note that $\chi(G) = 1$ if and only if G is trivial. If not trivial, then $\chi(G) = 2$ if and only if G is bipartite. It is clear that if $H \subseteq G$, then $\chi(H) \leq \chi(G)$. In particular, if G is disconnected, then $\chi(G)$ is simply the largest chromatic number among the components of G.

Example. We claim that $\chi(C_5) = 3$. To prove it, first we show that 3 colors are sufficient, e.g., we color the vertices blue, red, blue, red, yellow as we go counter-clockwise along the cycle. Second, we show that 2 colors do not suffice, e.g., since odd cycle is not bipartite.

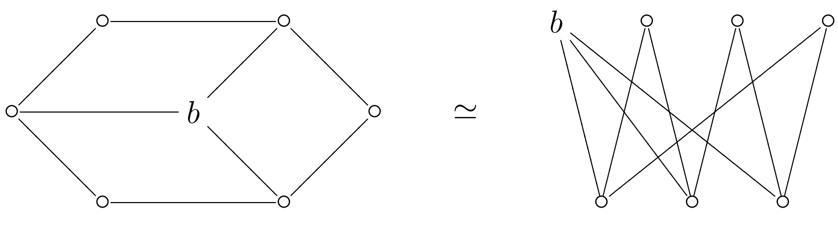
Algorithm 28 (Bi-Coloring)**.** Given a graph G, we apply two colors to determine bipartite or not bipartite and if so, we produce the bipartition subsets.

1. Any vertex can be chosen to start with by assigning the color black.

2. Let S denote the set of vertices which have already been colored. Now color every uncolored vertex in $N(S)$ white or black as determined by their adjacent color. If this is not possible, then G is not bipartite.

3. Repeat until all have been colored, in which case the vertices are bipartitioned according to their colors, black or white.

Example. We choose vertex b of the given graph and successfully color all vertices in the end. Omitting details, the bipartite graph with its bipartitions is then reconstructed as shown on the right.



Theorem 29. Let $\chi(G) = k$. Then the graph G has a vertex v such that $\deg(v) \geq k-1$, and at least k such vertices.

Proof. Let H be a minimal subgraph of G with $\chi(H) = k$, i.e., such that any subgraph of H will have chromatic number less than k. It is clear that H has at least k vertices. To complete the proof, we claim that in H we have $\deg(v) \geq k-1$ for all $v \in V_H$. This is true because by the choice of H, the subgraph of H minus a vertex v can be colored with $k-1$ colors. So if $\deg(v) \leq k-2$, then we can assign v one of these $k-1$ colors to complete coloring H. In that case $\chi(H) \leq k-1$, a contradiction. $\triangledown$

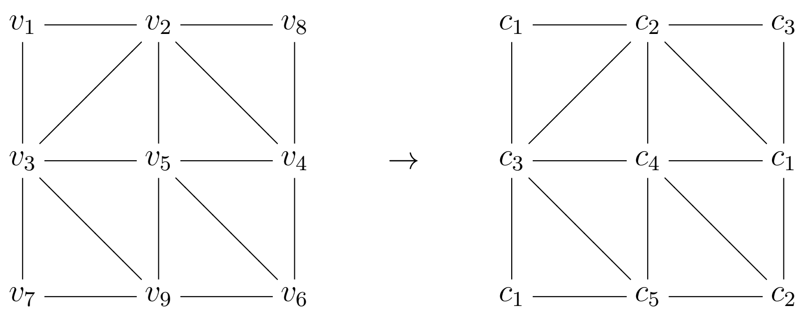
Theorem 30. For any graph G, we have $\chi(G) \leq \Delta(G) + 1$.

Proof. By the theorem, there is $v \in V_G$ such that $\Delta(G) \geq \deg(v) \geq \chi(G) - 1$. $\triangledown$

Algorithm 31 (Sequential Coloring)**.** We color a graph G with pre-ordered vertices.

1. Let $V_G = \{v_1, v_2, \ldots, v_n\}$ and we prepare the colors $c_1, c_2, \ldots, c_n$.
2. For $i = 1, 2, \ldots, n$, assign to v_i the color c_m with the least possible value of m.

Example. The graph G below has 9 ordered vertices. The sequential coloring algorithm yields the color sequence $(1, 2, 3, 1, 4, 2, 1, 3, 5)$. These 5 colors may not be minimum, so $\chi(G) \leq 5$ but the algorithm does not determine the chromatic number. Also note that a different vertex ordering will probably result in a different number of colors.



Algorithm 32 (Welsh-Powell Coloring)**.** We color the graph G by prioritizing vertices of larger degrees.

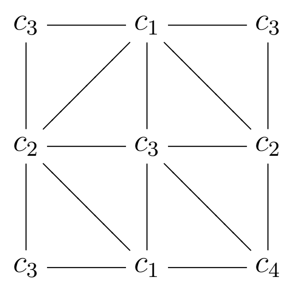
1. Order the vertices $w_1, w_2, \ldots, w_n$, from the largest degree to the smallest.
2. For $i = 1, 2, \ldots, n$, assign to w_i the color c_1 whenever possible.
3. For $i = 1, 2, \ldots, n$, assign to w_i the color c_2, if yet uncolored, whenever possible, and repeat in like manner using $c_3, c_4, \ldots$ until all have been colored.

Example. Using Welsh-Powell algorithm, we re-color the same graph from the previous example, showing the vertex re-ordering and its color assignment below:

v_i	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
$\deg(v_i)$	2	5	5	4	5	3	2	2	4

w_i	v_2	v_3	v_5	v_4	v_9	v_6	v_1	v_7	v_8
$\text{color}(w_i)$	c_1	c_2	c_3	c_2	c_1	c_4	c_3	c_3	c_3

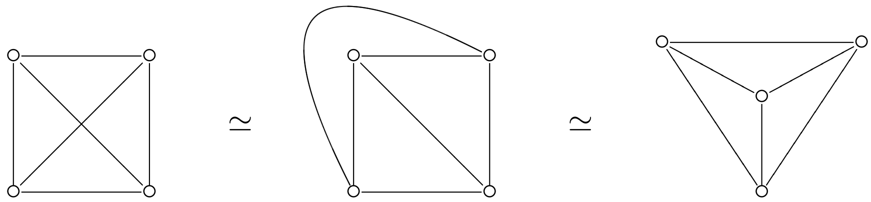
$\to$



The resulting color sequence is $(3, 1, 2, 2, 3, 4, 3, 3, 1)$. We have used only 4 colors this time, but again, in general no coloring algorithm is known to determine $\chi(G)$.

Definition. A graph is *planar* if it can be drawn in the plane such that no edges are crossing each other. This particular drawing of a planar graph is called a *plane graph*.

Example. We show two ways of drawing the graph K_4 without crossing edges:

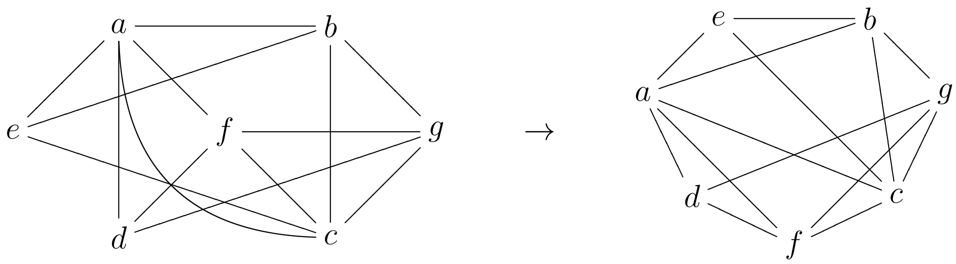


Similarly, we can show that $K_{2,3}$ is planar and later that K_5 and $K_{3,3}$ are not. Note that if G is disconnected, then G is planar if and only if each component is planar.

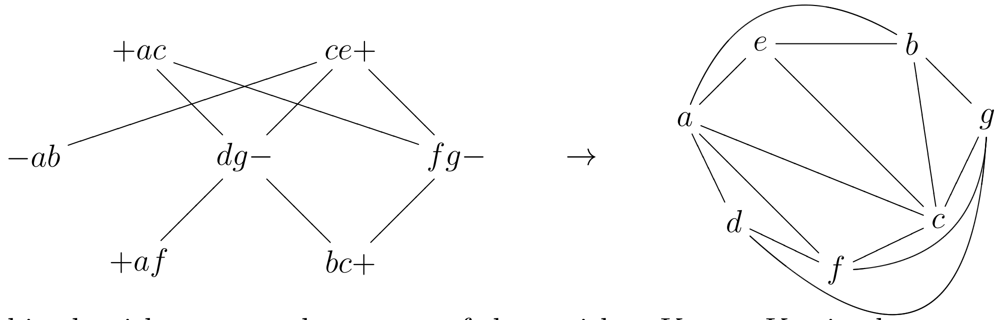
Algorithm 33 (Planarity Test for Hamilton Graphs)**.** Given a Hamilton graph G, we determine planar or not planar.

1. Draw a Hamilton cycle H by positioning the vertices in such a way that H is somewhat circular in shape.
2. Transform H into G by adding the remaining edges $e_1, e_2, \ldots, e_m$, all of which are drawn interior to the cycle H.
3. Let K be the graph with vertices $V_K = \{e_1, e_2, \ldots, e_m\}$ such that e_i, e_j are adjacent in K if and only if they are crossing each other in H.
4. Determine whether K is bipartite or not bipartite. The conclusion: G is planar if and only if K is bipartite.

Example. The given graph G is on the left. We then reconstruct G after the Hamilton cycle $aebgcfda$ shown on the right:



The resulting graph K below is shown on the left, which we prove bipartite by 2-coloring, represented by (+) and (–) colors. We then construct the plane graph G by moving the (–) edges exterior of the cycle while keeping the (+) edges interior:



With this algorithm we can have a proof that neither K_5 nor $K_{3,3}$ is planar.

Definition. A plane graph partitions the plane into subsets which are called *regions.* In other words, regions refer to the bounded areas interior to the plane graph, plus one unbounded exterior.

Example. The plane graph of K_4, pictured earlier, has four regions.

Theorem 34 (Euler's Formula)**.** Suppose that a connected plane graph has V vertices, E edges, and R regions. Then

$$V + R = E + 2$$

Proof. If the graph is a tree, then $E = V - 1$ and $R = 1$ as there is no bounded region. Hence, the identity $V + R = E + 2$ holds in a tree. A graph that is not a tree can be constructed by adding edges to its spanning tree, where adding one new edge would create one more bounded region, thereby preserving Euler's formula. $\triangledown$

Theorem 35 (Euler's Planarity Test)**.** Suppose that G is connected and that $V \geq 3$.

1. If $E > 3V - 6$, then G is not planar.
2. If $E > 2V - 4$ and G has no triangles, then G is not planar.

Proof. Given a fixed number of edges, the number of regions is maximized when every region is the interior of a triangle. (With higher polygons, a diagonal edge can be added while maintaining planarity.) Since every edge borders two regions, this maximum R is given by the relation $R_{\max} = \frac{2E}{3}$. Then, by Euler's formula,

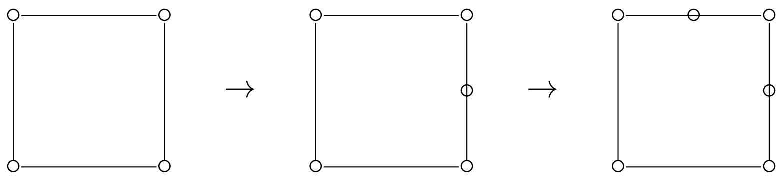
$$E = V + R - 2 \leq V + \frac{2E}{3} - 2$$

which simplifies to $E \leq 3V - 6$. This inequality holds for planar graphs, hence Euler's planarity test is justified. In cases where G has no triangles, we will have $R_{\max} = \frac{2E}{4}$, and the claim will follow similarly. $\triangledown$

Example. The graph K_5 has 10 edges, where $10 > 3(5) - 6$, hence not planar. As for $K_{3,3}$, we have 9 edges, 6 vertices, and no triangles (bipartite has no odd cycle), hence not planar since $9 > 2(6) - 4$.

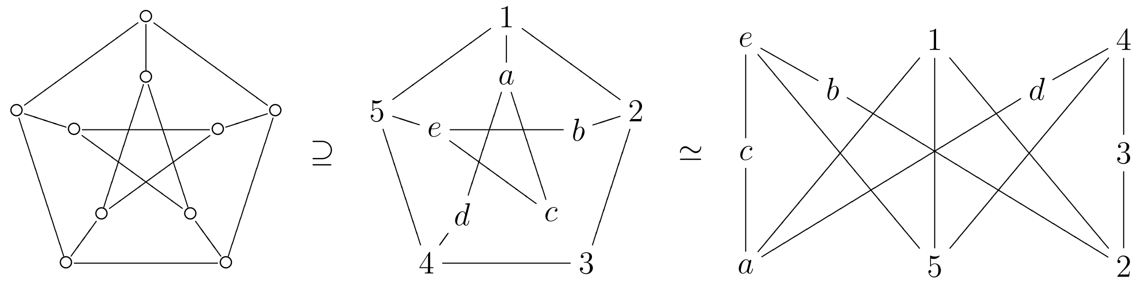
Definition. If we replace any edge $ab \in E_G$ by the path $\{av, vb\}$, where v is a new vertex, then the resulting graph is said to be *homeomorphic* to G. Now *homeomorphism* is an equivalence relation among graphs in which two of them are homeomorphic if one can be obtained from the other by iterating a finite number of replacements in this manner.

Example. We sketch below how to obtain C_6 by applying the procedure twice to C_4. In this way, it is not hard to see that any two cycles are homeomorphic.



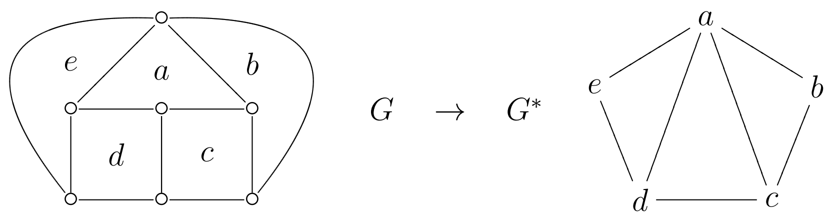
Theorem 36 (Kuratowski's Theorem)**.** A graph is planar if and only if it contains no subgraph homeomorphic to K_5 or $K_{3,3}$.

Example. The so-called *Peterson graph* given on the left is not planar. We prove this fact after finding a subgraph which, after a little modification in the way it is drawn, is shown to be homeomorphic to $K_{3,3}$:



Definition. The *dual graph* G^* of a plane graph G is the graph whose vertices are the interior regions of G, and where $rs \in E_{G^*}$ if and only if the regions r and s are bounded by a common edge in G.

Example. We show below a plane graph G with five interior regions, next to its dual graph G^* with these five as vertices:



As another example, one may draw and see that the dual graph of K_4 is K_3.

Theorem 37. The dual graph of any plane graph is planar. Conversely, every planar graph is the dual graph of some plane graph.

Proof. For each interior region r of the plane graph G, put the vertex r for the dual graph G^* right inside this region. Now between two adjacent vertices $r, s \in V_{G^*}$, the edge rs can be drawn as a curve that crosses the edge of G bordering the regions r and s and only this edge. This way we observe that the edges in G^* can be drawn without crossing one another, and that this procedure can also be done in reverse. $\triangledown$

Definition. A plane graph somewhat looks like a world map in which the interior regions represent countries. We say two countries are neighbors when they share a common edge boundary. (Having a common vertex does not make a neighbor.) In fact, map coloring was an early motivation for planar graphs.

We define the *chromatic number* of a map to be the least number of colors enough to color the countries such that neighboring countries have distinct colors. In other words, the chromatic number of a map G is given by $\chi(G^*)$.

Theorem 38. If G is planar, then $\chi(G) \leq 6$.

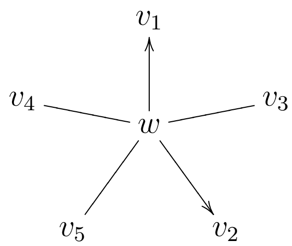
Proof. Suppose that G is planar with n vertices. We claim that there exists a vertex $w \in V_G$ such that $\deg(w) \leq 5$; If this were false, we would have $\deg G \geq 6n$ and then, by Euler's theorem, $|E_G| \geq 3n$, contradicting Euler's planarity test.

We proceed by induction on n. The result is trivial if $n \leq 6$, so we assume the theorem holds for all graphs of $n-1$ vertices. In particular, the subgraph of G obtained by removing the vertex w and its associated edges is planar, hence can be colored with 6 colors. Since $\deg(w) \le 5$, we may extend this coloring for all of G by assigning w one of the existing 6 colors, at least one that is not adjacent to it. That proves that $\chi(G) \le 6$. $\triangledown$

Theorem 39. If G is planar, then $\chi(G) \le 5$.

Proof. If G has 5 vertices or less, there is nothing to prove. Again by induction, we assume that G has n vertices and that the claim is true for any planar graph with less vertices than n. As in the preceding proof, G has a vertex w with $\deg(w) \le 5$. By the *color-degree* of a vertex we mean the number of distinct colors of vertices which are adjacent to it. Now because the subgraph G minus w can be colored with at most 5 colors, if the color-degree of w is 4 or less, then we easily assign to w one of the 5 colors to complete the inductive step.

The last case to consider is when $\deg(w) = 5$ with all five adjacent vertices, v_1, v_2, v_3, v_4, v_5, having distinct colors c_i. Without loss of generality, we have labeled these vertices such that v_3 is interior of the region bounded by the rays wv_1 and wv_2, while v_4 lies exterior of it.



Consider the subgraph $G_{1,2}$ of G consisting of all vertices which have been colored either c_1 or c_2, with their associated edges. Note that $v_1, v_2 \in G_{1,2}$. Suppose first that v_1 and v_2 belong to different components in $G_{1,2}$. In the component containing v_1, we swap c_1 and c_2. Doing so does not violate the rules of vertex coloring, but it does decrease the color-degree of w to 4 as v_1 and v_2 now have the same color, i.e., c_2.

But if v_1 and v_2 belong to the same component, then we have a path from v_1 to v_2 which becomes a cycle when combined with v_2w and wv_1. Since G is planar, this cycle must enclose either v_3 or v_4, but not both. Again, since G is planar, v_3 and v_4 then belong to different components in the subgraph $G_{3,4}$, defined in a similar way. This time, we decrease the color-degree of w by swapping c_3 and c_4 in the component containing v_3, and the proof is complete. $\triangledown$

Theorem 40 (The Four-Color Theorem)**.** The chromatic number of any map is at most four. Equivalently, the theorem states that if G is planar, then $\chi(G) \le 4$.

Exercises

1. (MC) Which graph has the largest number of edges?
   (a) K_{50} (b) $K_{50,50}$ (c) P_{100} (d) C_{200}

2. Determine (i) $\Delta(G)$ (ii) $\deg G$ (iii) $|E_G|$.
   (a) K_n (b) $K_{m,n}$ (c) P_n (d) C_n

3. A complete graph has 351 edges. Find the number of vertices.

4. A complete bipartite graph has degree 144. Find the least number of vertices.

5. (MC) Which graph is irregular?
(a) K_{99} (b) $K_{99,99}$ (c) P_{99} (d) C_{99}

6. Determine regular or irregular.
(a) K_n (b) $K_{m,n}$ (c) P_n (d) C_n

7. Draw a 3-regular graph of degree 18.

8. A graph is 5-regular with 35 edges. Find the number of vertices.

9. A complete bipartite graph is regular with 12 vertices. Find the number of edges.

10. Determine the degree sequence.
(a) K_n (b) $K_{m,n}$ (c) P_n (d) C_n

11. Determine graphical or not graphical. If graphical, draw it.
(a) $(3, 2, 2, 1, 1, 1)$ (b) $(4, 3, 3, 2, 1, 0)$ (c) $(5, 3, 2, 2, 1, 1)$ (d) $(5, 4, 4, 3, 3, 3, 3, 2, 2, 1)$

12. (MC) Which graph isomorphism is false?
(a) $K_{1,2} \simeq P_3$ (b) $K_{1,3} \simeq P_4$ (c) $K_{2,2} \simeq C_4$ (d) none of these

13. Draw two different graphs with the same degree sequence.
(a) $(4, 4, 3, 2, 2, 1)$ (b) $(2, 2, 2, 2, 2, 1, 1)$ (c) $(5, 3, 2, 2, 1, 1, 1, 1)$ (d) $(3, 3, 3, 3, 3, 3, 3, 3)$

14. (MC) Which graph does not contain C_4?
(a) K_5 (b) $K_{5,5}$ (c) $K_{3,5}$ (d) C_5

15. (MC) Which graph contains a bridge?
(a) K_9 (b) $K_{2,9}$ (c) P_9 (d) C_9

16. Determine if the graph may or may not contain a bridge.
(a) K_n (b) $K_{m,n}$ (c) P_n (d) C_n

17. Draw the complement of the graph.
(a) K_5 (b) $K_{3,4}$ (c) C_5 (d) P_5

18. Determine the complement of $K_{m,n}$ in general.

19. (MC) Which graph has disconnected complement?
(a) P_4 (b) C_6 (c) $K_{3,3}$ (d) none of these

20. Given the degree sequence in Problem 13, find the degree sequence for the complement of the graph.

21. Prove that $\overline{C_n}$ is connected if and only if $n \geq 5$.

22. Prove that if G is self-complementary, then $|E_G| = n(n-1)/4$, where $n = |V_G|$.

23. Give three different examples of self-complementary graphs.

24. Determine the adjacency matrix.
(a) K_5 (b) $K_{3,2}$ (c) P_5 (d) C_5

25. Let $V_G = \{v_1, v_2, \ldots, v_n\}$ with adjacency matrix A. Prove that $[A^2]_{ii} = \deg(v_i)$.

26. Let A denote the adjacency matrix for C_4 and B the adjacency matrix for $K_{2,2}$. Find a permutation matrix P such that $PAP^T = B$.

27. Determine the incidence matrix.
(a) K_4 (b) $K_{3,2}$ (c) P_5 (d) C_5

28. Given the incidence matrix, find the adjacency matrix.
(a) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$

29. Determine the degree matrix.
(a) K_n (b) $K_{m,n}$ (c) P_n (d) C_n

30. Given the incidence matrix in Problem 28, find the degree matrix.

31. (MC) Which graph is a tree?
(a) K_9 (b) $K_{9,9}$ (c) $K_{9,1}$ (d) C_9

32. Determine a tree or not a tree.
(a) K_n (b) $K_{m,n}$ (c) P_n (d) C_n

33. Give three different examples of a tree with 6 vertices.

34. Prove that adding an edge to a tree will produce a cycle.

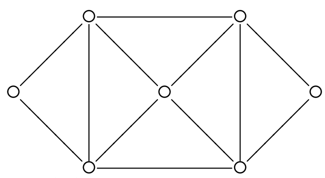
35. Prove that P_4 is the only self-complementary tree.

36. Determine if the graph may or may not contain a leaf.
(a) K_n (b) $K_{m,n}$ (c) P_n (d) C_n

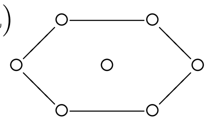
37. Let $G \in \{K_n, K_{m,n}, P_n, C_n\}$. Prove that G is a tree if and only if G has a leaf.

38. Let G have degree sequence $(d_1, d_2, \ldots, d_n)$ with $d_n \geq 1$. Prove that if G is a tree, then $d_1 + d_2 + \cdots + d_n = 2n - 2$. Conversely, prove that if $d_1 + d_2 + \cdots + d_n = 2n - 2$, then there exists a tree with such degree sequence, hence graphical.

39. (MC) Which one is a spanning tree of the given graph?



(a)



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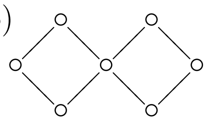

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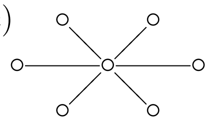

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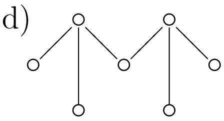
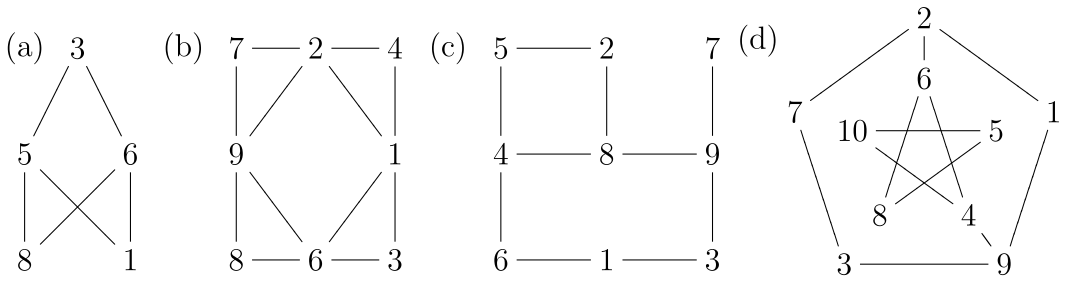


40. Determine the number of spanning trees using the Matrix-Tree theorem.
(a) K_4 (b) $K_{2,3}$ (c) P_4 (d) C_5

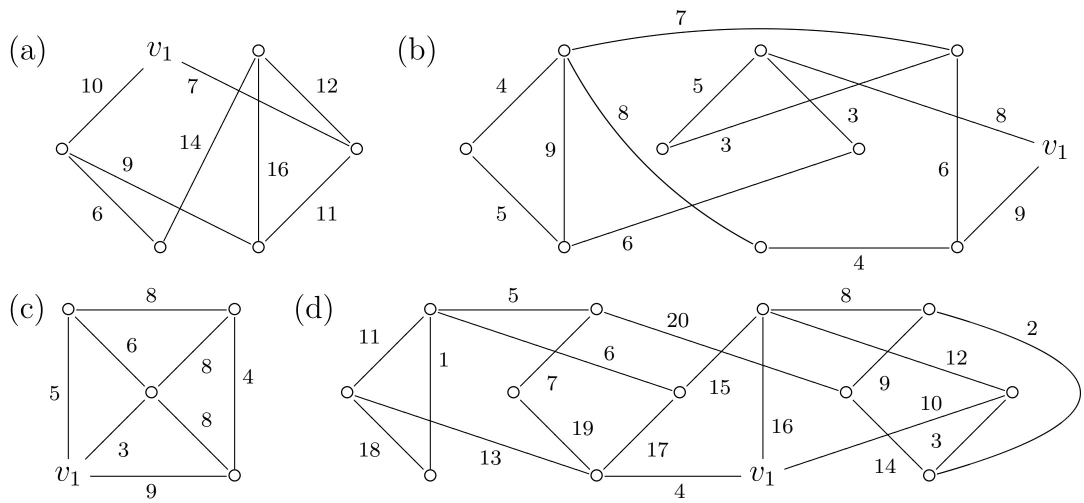
41. Prove that there are n^{n-2} different labeled trees with vertices $v_1, v_2, \ldots, v_n$, where $n \geq 2$, by showing that n^{n-2} is the number of spanning trees of a labeled K_n.

42. Apply the Depth-First Search algorithm starting at vertex number 1 and draw the (i) rooted (ii) unrooted spanning tree.



43. Repeat Problem 42 using the Breadth-First Search algorithm.

44. Find the minimal spanning tree and its total weight using Prim's algorithm.



45. Repeat Problem 44 using Kruskal's algorithm.

46. Determine the number of triangles in the graph using adjacency matrix.
(a) K_4 (b) $K_{2,2}$ (c) $K_{2,3}$ (d) K_5

47. Determine the number of triangles in K_n without using adjacency matrix.

48. Determine the distance matrix.
(a) K_5 (b) $K_{3,2}$ (c) P_5 (d) C_5

49. Draw two non-isomorphic graphs with the same distance matrix.

50. Given the incidence matrix in Problem 28, find the diameter of the graph.

51. (MC) Which graph has the largest diameter?
(a) K_{99} (b) $K_{99,99}$ (c) P_{99} (d) C_{99}

52. Determine the diameter.
(a) K_n (b) $K_{m,n}$ (c) P_n (d) C_n

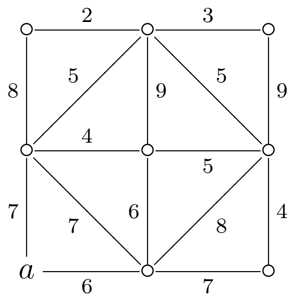
53. Draw a graph with 7 vertices, diameter 4, and no leaf.

54. Prove that if G is self-complementary, then $d(G) = 2$ or 3.

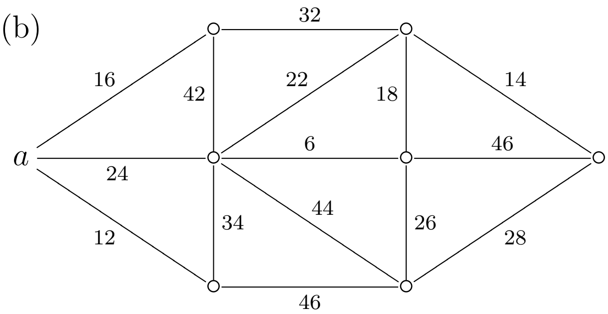
55. Prove that $d(\overline{C_n}) = 2$ if and only if $n \geq 5$.

56. Evaluate $d(a, x)$ using Dijkstra's algorithm.

(a)

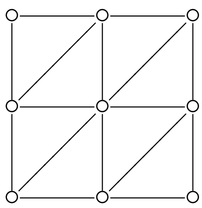


(b)



57. Find an Euler walk or Euler circuit in the graph, if any.

(a)



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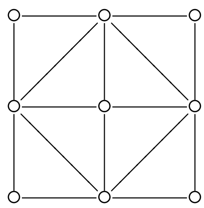

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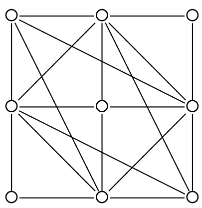

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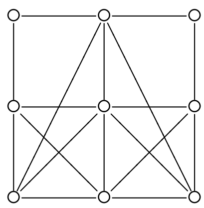


58. (MC) Which graph has an Euler walk but not Euler circuit?
(a) K_{99} (b) K_{100} (c) $K_{100,2}$ (d) $K_{99,2}$

59. Determine if the graph may or may not have an Euler walk or Euler circuit.
(a) K_n (b) $K_{m,n}$ (c) P_n (d) C_n

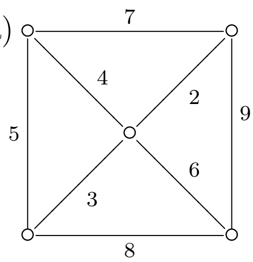
60. Prove that P_n is the only tree that has an Euler walk.

61. Determine all the values of n for which $\overline{C_n}$ has an Euler walk.

62. Let p be a prime number. Prove that if a connected graph is regular with p edges, then the graph is C_p.

63. Solve the Chinese Postman problem.

(a)



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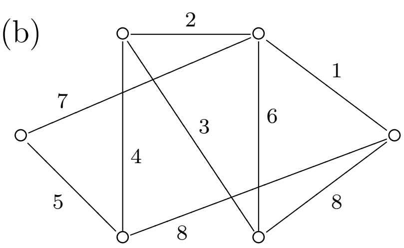

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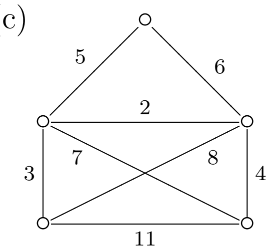

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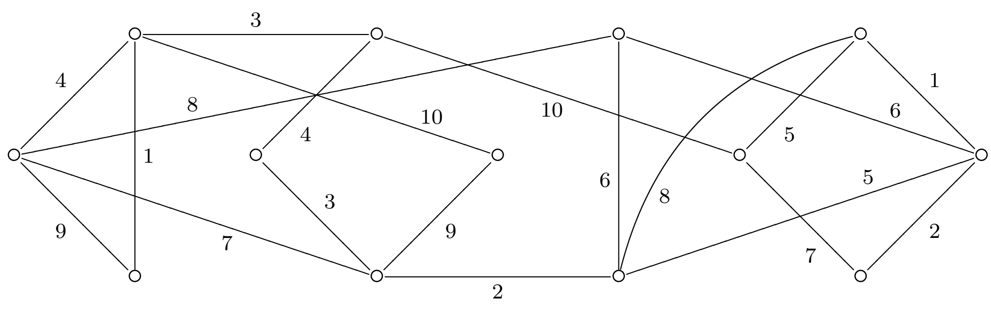
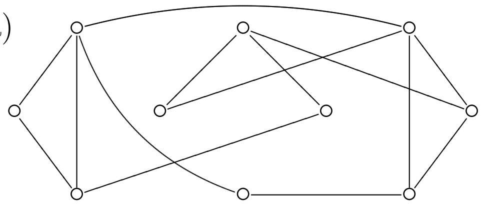


64. (MC) Which graph is not a Hamilton graph?
    (a) K_{99} (b) $K_{99,99}$ (c) $K_{100,99}$ (d) C_{99}

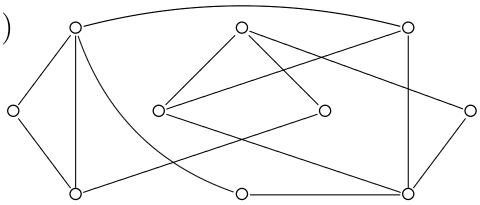
65. Determine Hamilton or not Hamilton.
    (a) K_n (b) $K_{m,n}$ (c) P_n (d) C_n

66. Determine Hamilton or not Hamilton with the help of degree-2 vertices.

(a)



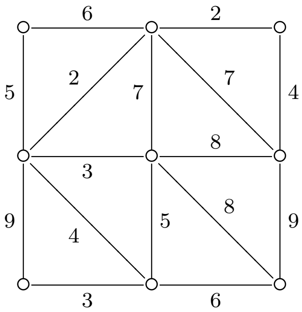
(b)



67. Prove that $\overline{C_n}$ is a Hamilton graph if and only if $n \geq 5$.

68. Solve the Traveling Salesman problem by drawing all possible Hamilton cycles.

(a)



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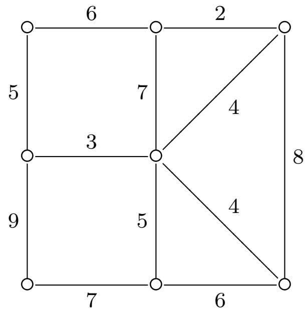

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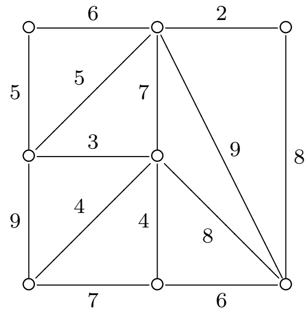

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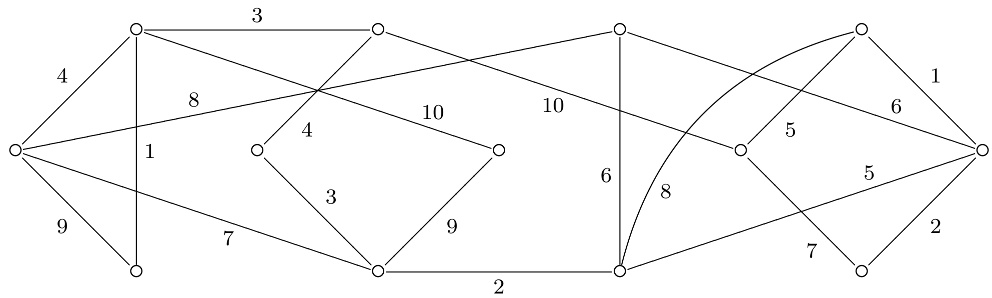


69. (MC) Which graph is not bipartite?
    (a) K_9 (b) $K_{8,8}$ (c) P_7 (d) C_6

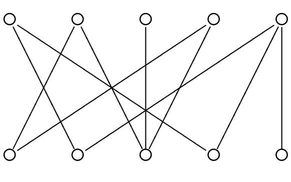
70. Determine bipartite or not bipartite.
    (a) K_n (b) $K_{m,n}$ (c) P_n (d) C_n

71. Let G be a bipartite graph with n vertices. Prove that (a) $|E_G| \leq n^2/4$ and (b) $|E_G| = n^2/4$ if and only if $G \simeq K_{n/2,n/2}$.

72. Prove that $\overline{C_n}$ is not bipartite if and only if $n \geq 5$.

73. Find a complete matching or prove not exist using Hall's theorem.

(a)



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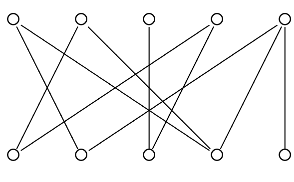

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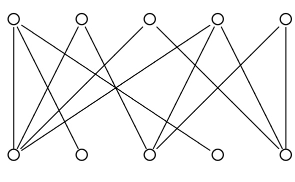


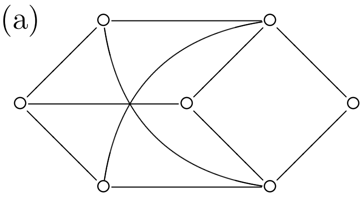
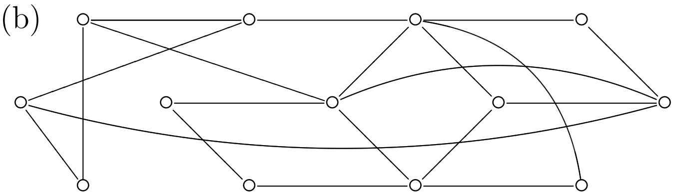
74. Let G be a bipartite graph with $V_G = X \sqcup Y$. Prove that if G is a Hamilton graph, then $|X| = |Y|$.

75. Prove that $K_{m,n}$ has a perfect matching if and only if it is also a Hamilton graph.

76. Draw all possible regular bipartite graphs with up to 6 vertices.

77. Let G be a connected graph. Prove that G is a 2-regular bipartite graph if and only if G is a cycle of even length.

78. Determine bipartite or not using bi-coloring. If bipartite, reconstruct the graph to show the bipartition subsets. If not bipartite, find an odd cycle in the graph.

(a)  (b) 

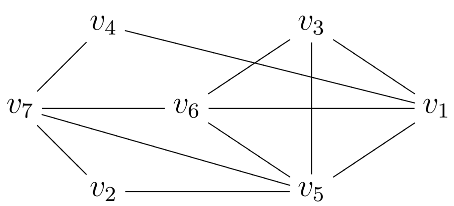
79. (MC) Which graph has the largest chromatic number?
(a) P_{99} (b) C_{99} (c) P_{100} (d) C_{100}

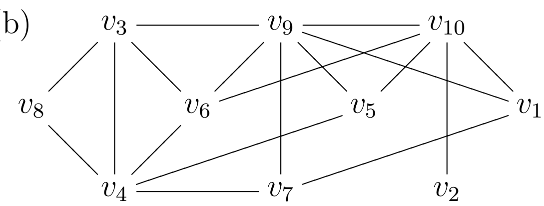
80. Determine the chromatic number for the graphs given in Problem 57.

81. Evaluate $\chi(\overline{C_6})$.

82. Determine if $\chi(G) = \Delta(G) + 1$ or $\chi(G) < \Delta(G) + 1$.
(a) K_n (b) $K_{m,n}$ (c) P_n (d) C_n

83. Color the graph using the Sequential Coloring algorithm.

(a) 

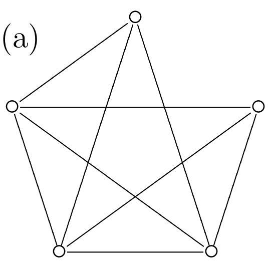
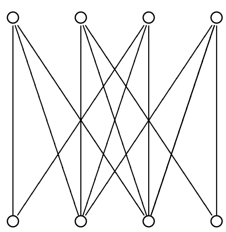
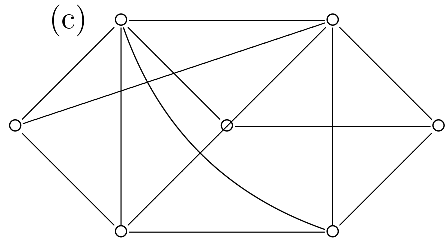
(b) 

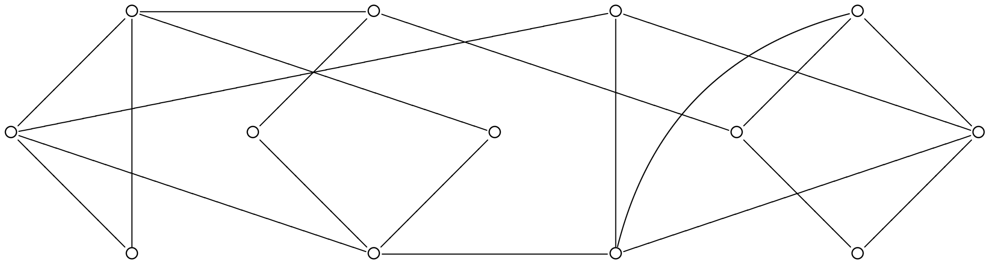
84. Repeat Problem 83 using Welsh-Powell algorithm.

85. Determine the chromatic number for the graphs given in Problem 83.

86. Show that in general $K_{2,n}$ is planar by drawing it.

87. Determine planar or not planar using the planarity test for Hamilton graphs. If planar, draw the plane graph.

(a)  (b)  (c) 

(d) 

88. Determine the number of regions in a plane graph with $|V_G| = 10$ and $\deg G = 32$.

89. Draw a plane graph with 8 vertices and 8 regions.

90. A bipartite plane graph has degree 42. Find the maximum number of regions.

91. Prove that $\overline{C_n}$ is not planar if and only if $n \geq 7$.

92. Prove that a d-regular graph is not planar for all $d \geq 6$.

93. Prove that if G is a planar graph with at least 11 vertices, then $\overline{G}$ is not planar.

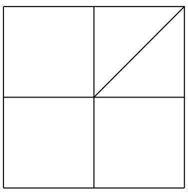
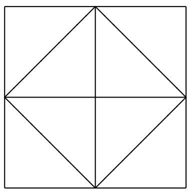
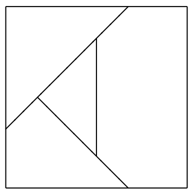
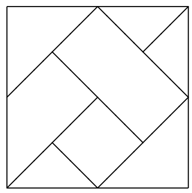
94. (MC) Which graph is not homeomorphic to any other?
(a) K_3 (b) K_4 (c) $K_{2,2}$ (d) C_9

95. Prove that every graph G is homeomorphic to a bipartite graph, if we replace every edge in G by a path of length two.

96. (MC) Which graph is the dual graph of $K_{2,9}$?
(a) K_9 (b) P_8 (c) $K_{8,1}$ (d) C_9

97. Draw a map with the given dual graph.
(a) K_4 (b) P_5 (c) $K_{2,3}$ (d) C_5

98. Determine the chromatic number of the map by coloring its dual graph.

(a)  (b)  (c)  (d) 

99. For the Peterson graph, given in the example following Theorem 36, determine
    (a) the degree sequence
    (b) the degree
    (c) regular or irregular
    (d) the diameter
    (e) Euler or not Euler
    (f) bipartite or not bipartite
    (g) the chromatic number
    (h) all the values of n such that C_n is a subgraph

100. The Peterson graph is neither planar nor Hamilton. What about its complement?