Homework 1 — Complex Analysis

1. Let $z = 5 + 4i$ and $w = 2 - i$. Evaluate using rectangular form.

   (a) $z^2 - w^2$ (b) $\bar{z}(z - iw)$ (c) zw^{-1} (d) $\overline{w} \div iz$

2. Find two complex numbers z such that $z^2 = 6i - 8$, by substituting $z = x + yi$.

3. Solve for $z \in \mathbb{C}$ using the quadratic formula.

   (a) $z^2 + 2z + 2 = 0$ (b) $iz^2 + 3z + 4i = 0$ (c) $3z^2 - 10iz - 3 = 0$

4. Draw the region in the complex plane with the given condition.

   (a) $|z - 2| \leq 4$ (b) $|2z + 3i| = 5$ (c) $\operatorname{Im} z > 1$ (d) $\operatorname{Re}(z^2) \geq 0$ (e) $\operatorname{Re}(\frac{1}{z}) < \frac{1}{2}$

Homework 2 — Complex Analysis

1. Convert to polar form.

   (a) $3i$ (b) $-2 + 2i\sqrt{3}$ (c) $-5 - 5i$ (d) $\sqrt{3} - i$

2. Convert to rectangular form.

   (a) $(1, 0)$ (b) $(2, -\pi/2)$ (c) $(3, \pi/6)$ (d) $(1, 2\pi/3)$

3. Evaluate by converting first to polar form then back to rectangular form.

   (a) $(1 + i)^6$ (b) $(1 + i\sqrt{3})^{10}$ (c) $(1 + i\sqrt{3})^{-2}$

4. Use polar form to find all complex solutions.

   (a) $z^2 = i$ (b) $z^8 = 1$ (c) $z^3 = -8$

Homework 3 — Complex Analysis

1. Evaluate e^z.

   (a) $z = 2 - 3\pi i$ (b) $z = 2 + 3\pi i$ (c) $z = (2 + \pi i)/4$

2. Find all numbers $z \in \mathbb{C}$ such that $e^z = w$.

   (a) $w = e$ (b) $w = -2$ (c) $w = i$ (d) $w = 1 + i\sqrt{3}$ (e) $w = -1 + i\sqrt{3}$

3. Prove that $\operatorname{Log}(1 + i)^2 = 2\operatorname{Log}(1 + i)$ but $\operatorname{Log}(-1 + i)^2 \neq 2\operatorname{Log}(-1 + i)$.

4. Evaluate using the principal Log.

   (a) i^i (b) $(-1)^{1/\pi}$ (c) $(1 - i)^{4i}$ (d) $(-\frac{e}{2} - i\frac{e\sqrt{3}}{2})^{3\pi i}$

Homework 4 — Complex Analysis

Prove the trigonometric identities:

1.	$\sin 2z = 2\sin z\cos z$	11.	$\sinh(z_1+z_2) = \sinh z_1\cosh z_2 + \cosh z_1\sinh z_2$
2.	$\cos 2z = \cos^2 z - \sin^2 z$	12.	$\cosh(z_1+z_2) = \cosh z_1\cosh z_2 + \sinh z_1\sinh z_2$
3.	$\sin^2 z + \cos^2 z = 1$	13.	$\sinh(z+2\pi i) = \sinh z$
4.	$\sin(z+\frac{\pi}{2}) = \cos z$	14.	$\cosh(z+2\pi i) = \cosh z$
5.	$\cos(z+\frac{\pi}{2}) = -\sin z$	15.	$\sinh z = \sinh x\cos y + i\cosh x\sin y$
6.	$\sin(z+\pi) = -\sin z$	16.	$\cosh z = \cosh x\cos y + i\sinh x\sin y$
7.	$\cos(z+\pi) = -\cos z$	17.	$\lvert\sinh z\rvert^2 = \sinh^2 x + \sin^2 y$
8.	$\cosh(-z) = \cosh z$	18.	$\lvert\cosh z\rvert^2 = \sinh^2 x + \cos^2 y$
9.	$\sinh(-z) = -\sinh z$	19.	$\sinh z = 0 \iff z = n\pi i$
10.	$\cosh^2 z - \sinh^2 z = 1$	20.	$\cosh z = 0 \iff z = (\frac{\pi}{2}+n\pi)i$

Homework 5 — Complex Analysis

1. Prove using the definition of limit.

   (a) $\lim_{z\to 1-i}(2z+iz) = 3-i$ (b) $\lim_{z\to 2+i} 3z - 2iz = 8-i$

2. Prove that $\lim_{z\to 0}(z/\bar{z})^2$ does not exist by showing that the limits along the lines $y=0$ and $y=x$ are not equal.

3. Prove the limit involving the neigborhood of infinity.

   (a) $\lim_{z\to\infty}\frac{4z^2}{(z-1)^2} = 4$ (b) $\lim_{z\to 1}\frac{1}{(z-1)^3} = \infty$ (c) $\lim_{z\to\infty}\frac{z^2+1}{z-1} = \infty$

4. Find the real functions $u(x,y)$ and $v(x,y)$ such that $f(z) = u+iv$.

   (a) $f(z) = \bar{z}^2 + 2iz$ (b) $f = z^3+z+1$ (c) $f = e^{\bar{z}}$ (d) $f = e^{z^2}$ (e) $f = e^{1/z}$

5. Find the domain where $f(z)$ is continuous.

   (a) $f(z) = e^{1/z}$ (b) $f = \frac{\cos 2z}{\sin z}$ (c) $f = \operatorname{Log}(z+2)$ (d) $f = \frac{\sin^2\bar{z}}{z+\bar{z}}$ (e) $f = \frac{1}{z^3+4z}$

6. Evaluate the derivative.

   (a) $f(z) = 3z^2-2z+4$ (b) $f = (1-4z^2)^3$ (c) $f = \frac{z-1}{2z+1}$ (d) $f = \frac{(1+z^2)^4}{z^2}$

Homework 6 — Complex Analysis

Use Cauchy-Riemann equations to determine (a) the domain where $f'(z)$ exists and (b) the domain where $f(z)$ is analytic, and (c) find $f'(z)$.

1.	$z-\bar{z}$	5.	$z\operatorname{Im} z$	9.	$e^{-y}\sin x - ie^{-y}\cos x$
2.	$e^x e^{-iy}$	6.	$x^3+i(1-y)^3$	10.	$e^y e^{ix}$
3.	$e^{-x}e^{-iy}$	7.	$2x+ixy^2$	11.	$e^x(y^2+iy-3i)$
4.	x^2+iy^2	8.	$3x+y-ix+3iy$	12.	$2x^2-y^3+i(x+2xy-y^2)$

Homework 7 — Complex Analysis

1. Use the formula $\frac{d}{dz}(e^z) = e^z$ to evaluate $f'(z)$.
   (a) $f(z) = \cosh z$ (b) $f = \sinh z$ (c) $f = \tan z$ (d) $f = \cot z$

2. Use polar form to prove that $f'(z)$ exists in the given domain and find it.
   (a) $f(z) = 1/z^4 \quad (z \neq 0)$
   (b) $f(z) = \sqrt{r}e^{i\theta/2} \quad (r > 0, -\pi < \theta < \pi)$
   (c) $f(z) = e^{-\theta}\cos(\ln r) + ie^{-\theta}\sin(\ln r) \quad (r > 0, 0 < \theta < 2\pi)$

3. Prove that $u(x, y)$ is harmonic in some domain and find a harmonic conjugate $v(x, y)$.
   (a) $u(x, y) = 2x(1-y)$ (b) $u = 2x-x^3+3xy^2$ (c) $u = \sinh x \sin y$ (d) $u = \frac{y}{x^2+y^2}$

4. Prove for a given domain D.
   (a) If $f(z)$ and $\overline{f(z)}$ are both analytic in D, then f is constant in D.
   (b) If $f(z)$ is analytic in D and $|f(z)|$ is constant in D, then f is constant in D.
   (c) If $v(x, y)$ is a harmonic conjugate of $u(x, y)$ in D and conversely u is a harmonic conjugate of v in D, then both u and v are constant in D.
   (d) In the same domain, v is a harmonic conjugate of u if and only if $-u$ is a harmonic conjugate of v.

Homework 8 — Complex Analysis

1. Evaluate the line integrals.
   (a) $\int_0^1 (1 + it)^2\, dt$ (b) $\int_1^2 (\frac{1}{t} - i)^2\, dt$ (c) $\int_0^{\pi/6} e^{i2t}\, dt$

2. Determine $z = z(t)$ and the interval $t \in [a, b]$ for each contour C.
   (a) circle radius 2 centered at $2 - 3i$
   (b) straight line from $-1 + 2i$ to $1 - 2i$
   (c) curve from 0 to $2 + 4i$ along the parabola $y = x^2$
   (d) upper semi-circle radius 1 centered at 1
   (e) straight line from -1 to $2 + 6i$

3. Evaluate the integral $\int_C f(z)\, dz$, where C is the semi-circle $z(t) = e^{it}$ $(0 \leq t \leq \pi)$.
   (a) $f(z) = \frac{z+2}{z}$ (b) $f = \bar{z} - 1$ (c) $f = z^2 + z$ (d) $f = z^i$ (principal Log)

4. Evaluate $\int_C \pi e^{\pi\bar{z}}\, dz$ where C is the square in the complex plane with vertices $\{0, 1, 1 + i, i\}$ and positive orientation.

5. Evaluate the contour integral $\int_C f(z)\, dz$ where

$$f(z) = f(x + yi) = \begin{cases} 1 & \text{if } y < 0 \\ 4y & \text{if } y > 0 \end{cases}$$

   and C is the arc from $-1 - i$ to $1 + i$ along the curve $y = x^3$.

Homework 9 — Complex Analysis

Evaluate using anti-derivative when possible, or using definition otherwise.

1. $\int_C e^{\pi z}\,dz$, where C is the straight line from i to $i/2$.
2. $\int_C \bar{z}\,dz$, where C is the circle $z(t) = i + e^{it}$ $(0 \le t \le 2\pi)$.
3. $\int_C \cos\frac{z}{2}\,dz$, where C is the straight line from 0 to π followed by another straight line from π to $\pi + 2i$.
4. $\int_C z^{-2}\,dz$, where C is the circle $z(t) = 3e^{it}$ $(-\pi \le t \le \pi)$.
5. $\int_C (z-2)^3\,dz$, where C is the semi-circle $z(t) = 2 + e^{it}$ $(0 \le t \le \pi)$.
6. $\int_C \frac{1}{z}\,dz$, where C is the semi-circle $z(t) = 2e^{it}$ $(\frac{\pi}{2} \le t \le \frac{3\pi}{2})$.

Homework 10 — Complex Analysis

Evaluate using the Cauchy-Goursat Theorem or the Cauchy Integral Formula. The contour C is the square bounded by $x = \pm 2$ and $y = \pm 2$.

1. $\displaystyle\int_{|z|=1} \frac{z^2}{z-3}\,dz$
2. $\displaystyle\int_C \frac{e^{-z}}{z-\frac{\pi i}{2}}\,dz$
3. $\displaystyle\int_{|z|=3} \frac{2z^2-z-2}{z+4i}\,dz$
4. $\displaystyle\int_C \frac{\cos z}{z(z^2+8)}\,dz$
5. $\displaystyle\int_{|z-i|=2} \frac{1}{z^2+4}\,dz$
6. $\displaystyle\int_{|z|=1} ze^{-z}\,dz$
7. $\displaystyle\int_{|z|=1} \mathrm{Log}(z+2)\,dz$
8. $\displaystyle\int_C \frac{z}{2z+1}\,dz$
9. $\displaystyle\int_{|z|=5} \frac{z^3+2z}{(z+3)^3}\,dz$
10. $\displaystyle\int_{|z|=1} \frac{1}{z^2+2z+2}\,dz$
11. $\displaystyle\int_{|z|=2} \frac{1}{z^2+2z+2}\,dz$
12. $\displaystyle\int_C \frac{\cosh z}{z^4}\,dz$
13. $\displaystyle\int_{|z-i|=2} \frac{1}{(z^2+4)^2}\,dz$
14. $\displaystyle\int_{|z|=2} \frac{1}{(z^2+1)^2}\,dz$
15. $\displaystyle\int_{|z-2i|=3} \frac{1}{z^3+4z}\,dz$

Homework 11 — Complex Analysis

Evaluate the real definite integrals using Cauchy Integral Formula.

1. $\displaystyle\int_0^{2\pi} \frac{dx}{2+\cos x}$
2. $\displaystyle\int_{-\pi}^{\pi} \frac{dx}{5+4\sin x}$
3. $\displaystyle\int_0^{2\pi} \frac{\cos 2x\,dx}{5-4\cos x}$
4. $\displaystyle\int_0^{2\pi} \frac{dx}{3-2\cos x+\sin x}$
5. $\displaystyle\int_0^{2\pi} \frac{dx}{5-3\sin x}$
6. $\displaystyle\int_0^{2\pi} \frac{dx}{(5-3\sin x)^2}$

Homework 12 — Complex Analysis

Evaluate the improper real integrals using Cauchy Integral Formula.

1. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$
2. $\displaystyle\int_0^{\infty} \frac{dx}{(1+x^2)^2}$
3. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2dx}{(x^2+1)(x^2+4)}$
4. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^3}$
5. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$
6. $\displaystyle\int_{-\infty}^{\infty} \frac{xdx}{(x^2+1)(x^2+2x+2)}$

Homework 1

1. (a) $6+24i$ (b) $12+10i$
   (c) $\frac{1}{5}(6+13i)$ (d) $\frac{1}{41}(-3-14i)$
2. $\pm(1+3i)$
3. (a) $-1\pm i$ (b) $-i, 4i$ (c) $i/3, 3i$
4. (a) on and inside circle radius 4 center $(2,0)$ (b) circle radius 2.5 center $(0,-1.5)$ (c) half-plane above $y=1$ (d) bounded by the lines $y=\pm x$ (e) inside circle radius 1 center $(1,0)$

Homework 2

1. (a) $(3, \pi i/2)$ (b) $(4, 2\pi i/3)$
   (c) $(5\sqrt{2}, -3\pi i/2)$ (d) $(2, -\pi i/6)$
2. (a) 1 (b) $-2i$ (c) $\frac{3}{2}(\sqrt{3}+i)$ (d) $-\frac{1}{2}+\frac{i}{2}\sqrt{3}$
3. (a) $-8i$ (b) $-512-512i\sqrt{3}$ (c) $-\frac{1}{8}-\frac{i}{8}\sqrt{3}$
4. (a) $(1,\pi/4),(1,-3\pi/4)$ (b) $(1,k\pi/4), 0\leq k\leq 7$ (c) $(2,(2k+1)\pi/3), k=0,1,2$

Homework 3

1. (a) $-e^2$ (b) $-e^2$ (c) $\sqrt{e/2}(1+i)$
2. (a) $1+2n\pi i$ (b) $\ln 2+(2n+1)\pi i$
   (c) $\frac{\pi}{2}i+2n\pi i$ (d) $\ln 2+\frac{\pi}{3}i+2n\pi i$
   (e) $\ln 2+\frac{2\pi}{3}i+2n\pi i$
3. $\ln 2+\pi i/2=\ln 2+\pi i/2$
   $\ln 2+3\pi i/2\neq \ln 2-\pi i/2$
4. (a) $e^{-\pi/2}$ (b) e^i
   (c) $e^{\pi}\cos\ln 4+ie^{\pi}\sin\ln 4$ (d) $-e^{2\pi^2}$

Homework 4

Use Theorem 2.4 for Problems 1–7 and Theorem 2.6 for Problems 8–20.

Homework 5

1. (a) Let $\delta=\epsilon/3$ (b) Let $\delta=\epsilon/5$
2. $1\neq -1$
3. Use Theorem 3.3
4. (a) $u=x^2-y^2-2y;\ v=2x-2xy$
   (b) $u=x^3-3xy^2+x+1;$
   $v=-y^3+3x^2y+y$
   (c) $u=e^x\cos y;\ v=-e^x\sin y$
   (d) $u=e^{x^2-y^2}\cos 2xy; v=e^{x^2-y^2}\sin 2xy$
   (e) $u=e^{x/(x^2+y^2)}\cos(y/(x^2+y^2));$
   $v=-e^{x/(x^2+y^2)}\sin(y/(x^2+y^2))$
5. (a) $\{z\neq 0\}$ (b) $\mathbb{C}\setminus\{\pi n \mid n\in\mathbb{Z}\}$
   (c) $\mathbb{C}\setminus(-\infty,-2]$ (d) $\{\operatorname{Im} z\neq 0\}$
   (e) $\{z\neq 0,\pm 2i\}$
6. (a) $6z-2$ (b) $-24z(1-4z^2)^2$
   (c) $3(2z+1)^{-2}$ (d) $2(4z^2-1)(1+z^2)^3z^{-3}$

Homework 6

(1) $\emptyset$ (2) $\emptyset$ (3) $\mathbb{C}\to f'=-e^{-x}e^{-iy}$
(4) $f'(x+xi)=2x$ (5) $f'(0)=0$
(6) $f'(i)=0$ (7) $\emptyset$ (8) $\mathbb{C}\to f'=3-i$
(9) $\mathbb{C}\to f'=e^{-y}e^{-ix}$ (10) $\emptyset$
(11) $f'(x+i)=e^x(1-2i)$
(12) $f'(\frac{1}{3},-\frac{1}{3})=\frac{1}{3}(4+i); f'(-1,1)=3i-4$

Homework 7

1. (a) $\sinh z$ (b) $\cosh z$ (c) $\cos^{-2} z$ (d) $-\sin^{-2} z$
2. (a) $f'=-4z^{-5}$ (b) $f'=\frac{1}{2f}$ (c) $f'=if/z$
3. (a) x^2-y^2+2y (b) $2y-3x^2y+y^3$
   (c) $-\cosh x\cos y$ (d) $x/(x^2+y^2)$
4. Use Theorem 4.4

Homework 8

1. (a) $\frac{2}{3}+i$ (b) $-\frac{1}{2}-i\ln 4$ (c) $\frac{1}{4}(\sqrt{3}+i)$
2. (a) $2-3i+2e^{it}$ $(0\leq t\leq 2\pi)$
   (b) $t-2it$ $(-1\leq t\leq 1)$
3. (a) $2\pi-2$ (b) $2+\pi i$
   (c) $-\frac{2}{3}$ (d) $\frac{1+e^{-\pi}}{2}(-1+i)$
4. $4e^{\pi}-4$
5. $2+3i$

Homework 9

(1) $\pi+\pi i$ (2) $2\pi i$ (3) $e+1/e$
(4) 0 (5) 0 (6) πi

Homework 10

(1) 0 (2) 2π (3) 0 (4) $\pi i/4$ (5) $\pi/2$ (6) 0
(7) 0 (8) $-\pi i/2$ (9) $-18\pi i$ (10) 0 (11) 0
(12) 0 (13) $\pi/16$ (14) π (15) $\pi i/4$

Homework 11

(1) $\pi/\sqrt{2}$ (2) $\pi/4$ (3) $\pi/3$
(4) $3\pi/256$ (5) π (6) $-\pi/5$

Homework 12

(1) $2\pi/\sqrt{3}$ (2) $2\pi/3$ (3) 3π
(4) π (5) $3\pi/2$ (6) $5\pi/32$

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